Interconnection of Dirac Structures and Lagrange–Dirac Dynamical Systems

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Abstract

Dirac structures unify both presymplectic and Poisson structures, which serve an implicit generalization of Lagrangian and Hamiltonian systems including the case of non-holonomic mechanics as well as the case of degenerate Lagrangians. It is known that a Dirac structure represents a power conserving interconnection structure between physical systems. In this paper, we investigate the interconnection of distinct Dirac structures and associated physical systems. First, we make a brief review on induced Dirac structures and Lagrange-Dirac dynamical systems. Second, we consider how distinct Dirac structures $D_1, ..., D_n$ can be interconnected through a Dirac structure D_{int}. To do this, we introduce a tensor product called the *bowtie product*, $\bowtie$, of Dirac structures and then show that the interconnection of Dirac structures can be given by $(D_1 \oplus \cdots \oplus D_n) \bowtie D_{\mathrm{int}}$. We also explore variational structures associated to the interconnection of Lagrange-Dirac systems. Lastly, we demonstrate the theory of interconnection of Dirac structures and associated Lagrange-Dirac dynamical systems by some examples including electric circuits, nonholonomic mechanical systems, and simple mass-spring mechanical systems.

Contents

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1 Introduction

A large class of physical and engineering problems can be described in terms of Lagrangian and Hamiltonian systems, such as controlled mechanical systems, externally forced and dissipative systems, collisions, electric circuits, stochastic systems, and field theories such as electromagnetism and elasticity. Such systems are usually categorized into Lagrangian or Hamiltonian systems with constraints, whether holonomic or nonholonomic; however, one may be faced with serious problems on ever larger scales becoming more and more heterogeneous, involving a mixture of mechanical and electrical components, with flexible and rigid parts, and elastic or magnetic couplings (see, for instance, Yoshimura [1995]; Bloch [2003] and Afshari, Bhat, Hajimiri and Marsden [2006]). Dirac structures are known as a powerful tool for providing a natural geometric framework for describing the structure of such diverse Lagrangian and Hamiltonian systems on the product of configuration spaces by introducing nontrivial constraints that model interactions between subsystems, which is called an *interconnection*. The notion of an interconnection was originally developed by Kron [1963] in his book of "Diakoptics" for modeling complicated physical systems. The word "diakoptics" denotes a procedure of *tearing* a dynamical system into disconnected subsystems as well as *interconnecting* them to reconstruct the original system. The original system may be regarded as an interconnected system of subsystems, with constraints induced from the interconnection. Interconnection has been studied through the lens of power conservation motivated by the bond graph theory established by Paynter [1961]. In electric circuit theory, the interconnection network has been widely employed through nonenergic multiports (see Brayton [1971]; Wyatt and Chua [1977]), where Kirchhoff's current law gives a constraint distribution on a configuration charge space such that Tellegen's theorem holds. In mechanics, it was shown by Yoshimura [1995] that kinematical constraints due to mechanical joints, coordinate transformations as well as force equilibrium conditions in d'Alembert principle can be represented by the interconnection modeled by nonenergic multiports using the bond graph theory in the context of Lagrangian mechanics.

It was demonstrated by van der Schaft and Maschke [1995] that the interconnection can be represented by Dirac structures induced from Poisson structures and also that nonholonomic systems and L-C circuits can be represented by *implicit Hamiltonian systems*. On the Lagrangian side, it was shown by Yoshimura and Marsden [2006a] that nonholonomic mechanical systems and L-C circuits (as degenerate Lagrangian systems) can be formulated by *implicit Lagrangian systems* associated with the Dirac structures on the cotangent bundles induced from given constraint distributions. From the view point of variational structures, it was shown by Yoshimura and Marsden [2006b] that the standard implicit Lagrangian systems, namely, *implicit Euler-Lagrange equations* for unconstrained systems can be derived from the *Hamilton-Pontryagin principle* and also that the Lagrange-Dirac dynamical systems can be formulated in the context of *Lagrange-d'Alembert-Pontryagin principle*.

From the viewpoint of control theory, a notion of *implicit port-controlled Hamiltonian (IPCH) systems* (an implicit Hamiltonian system with external control inputs) was developed by van der Schaft and Maschke [1995] (see also Bloch and Crouch [1997], Blankenstein [2000] and van der Schaft [1996]) and much effort has been devoted to the passivity based control for the interconnected mechanical systems (Ortega et al [1998]). For the case of regular Lagrangian systems, the equivalence between *controlled Lagrangian (CL) systems*

and *controlled Hamiltonian (CH)* systems was shown by Chang, Bloch, Leonard, Marsden, and Woolsey [2002] and the reduction theory for the CL systems was shown by Chang and Marsden [2003]. For the general case in which a given Lagrangian is degenerate, an implicit Lagrangian analogue of IPCH systems, namely, a notion of *implicit port-controlled Lagrangian (IPCL) systems* for electrical circuits was constructed by Yoshimura and Marsden [2006c] and Yoshimura and Marsden [2007a], where it was shown that L-C transmission lines can be represented in the context of the IPLC system by employing induced Dirac structures with Lagrange multipliers.

Recently, it was shown by Cervera, van der Schaft, and Banos [2007] that any power-conserving interconnection of IPCH systems is another port-Hamiltonian system, with the Dirac structure being the *composition* of the Dirac structures of its constituent parts, Hamiltonian the sum of the Hamiltonians, and resistive relations determined by the individual resistive relations. As a control application, the feedback interconnection of a plant port-Hamiltonian system with a controller port-Hamiltonian system can be represented by the composition of a plant Dirac structure with a controller Dirac structure.

Goals and Organizations of the Paper. One of the main goals in this paper is to develop a simple expression for the interconnection of distinct Dirac structures and their associated mechanical systems; namely, we will show how to obtain an interconnection of distinct Dirac structures $D_1, ..., D_n$ on distinct manifolds $M_1, ..., M_n$ as another Dirac structure D on $M_1 \times \cdots \times M_n$ by "interconnecting" $D_1, ..., D_n$. To do this, we will introduce two mathematical ingredients, namely, the *Dirac sum* $\oplus$ and the *bowtie product* $\bowtie$ of Dirac structures and we will show how two distinct Dirac structures can be interconnected to yield another Dirac structure via an *interconnection* Dirac structure (usually labeled D_{int} in this paper) under an assumption of transversal intersections. In particular, the interconnection of Dirac structures is $D = (D_1 \oplus \cdots \oplus D_n) \bowtie D_{\mathrm{int}}$. The interconnection Dirac structure D_{int} provides an interface to interconnect "distinct" Dirac structures $D_1, ..., D_n$ to yield another Dirac structure D. We will also show how the associated interconnection of Lagrange-Dirac dynamical systems can be developed in the context of variational structures. We will demonstrate our theory by some examples of circuits and mechanical systems.

This paper consists of the following parts:

1. We briefly review Dirac structures and we show how associated Lagrange-Dirac dynamical systems can be developed by the *Lagrange-d'Alembert-Pontryagin principle* (see Yoshimura and Marsden [2006b]).

2. We show how distinct Dirac structures can be interconnected to yield another Dirac structure on the product manifold through an interconnection Dirac structure. To do this, we develop a map called the Dirac sum $\oplus$ and a tensor product called the bowtie product $\bowtie$ between Dirac structures.

3. We explore the variational structure of interconnection of Lagrange-Dirac dynamical systems; namely, the interconnected Lagrange-Dirac dynamical system can be formulated in the context of the Lagrange-d'Alembert -Pontryagin principle.

4. We demonstrate some illustrative examples of circuits and nonholonomic systems, which can be generalized to more complex systems.

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2 Dirac Structures

Linear Dirac Structures. First, let us recall the definition of a linear Dirac structure, namely, a *Dirac structure on a vector space* V; we assume that V is finite dimensional for simplicity (see, Courant and Weinstein [1988]). Let V^* be the dual space of V, and $\langle \cdot \,, \cdot \rangle$ be the natural paring between V^* and V. Define the symmetric paring $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ on $V \oplus V^*$ by

$$\langle\!\langle \, (v, \alpha), (\bar{v}, \bar{\alpha}) \, \rangle\!\rangle = \langle \alpha, \bar{v} \rangle + \langle \bar{\alpha}, v \rangle,$$

for $(v, \alpha), (\bar{v}, \bar{\alpha}) \in V \oplus V^*$.

A *Dirac structure* on V is a subspace $D \subset V \oplus V^*$ such that $D = D^{\perp}$, where $D^{\perp}$ is the orthogonal of D relative to the pairing $\langle\!\langle \cdot, \cdot \rangle\!\rangle$.

Dirac Structures on Manifolds. Let M be a smooth manifold[1] and let $TM \oplus T^*M$ denote the Whitney sum bundle over M, namely, the bundle over the base M and with fiber over the point $x \in M$ equal to $T_xM \times T_x^*M$. A subbundle $D \subset TM \oplus T^*M$ is called an almost Dirac structure on M, when $D(x)$ is a Dirac structure on the vector space T_xM at each point $x \in M$. A given two-form Ω on M together with a distribution Δ on M determines an *(almost) Dirac structure* on M as follows: For each $x \in M$,

$$\begin{aligned} D(x) = \{(v, \alpha) \in T_xM \times T_x^*M \;\mid v \in \Delta(x), \text{ and} \\ \alpha(w) = \Omega_{\Delta}(x)(v, w) \; \text{ for all } \; w \in \Delta(x)\}, \end{aligned} \tag{2.1}$$

where Ω_{Δ} is the restriction of Ω to Δ. We can also restate the Dirac structure as, for each $x \in M$,

$$D = \{(v, \alpha) \in T_xM \times T_x^*M \;\mid\; v \in \Delta(x) \text{ and } \Omega^{\flat}(x)v - \alpha \in \Delta^{\circ}(x)\},$$

where Δ° is an annihilator of Δ.

We call D an *integrable Dirac structure* if the following integrability condition

$$\langle \pounds_{X_1} \alpha_2, X_3 \rangle + \langle \pounds_{X_2} \alpha_3, X_1 \rangle + \langle \pounds_{X_3} \alpha_1, X_2 \rangle = 0 \tag{2.2}$$

is satisfied for all pairs of vector fields and one-forms (X_1, α_1), (X_2, α_2), (X_3, α_3) that take values in D, where $\pounds_X$ denotes the Lie derivative along the vector field X on M.

Remark. Let $\Gamma(TM \oplus T^*M)$ be a space of local sections of $TM \oplus T^*M$, which is endowed with the skew-symmetric bracket $[\ ,\ ] : \Gamma(TM \oplus T^*M) \times \Gamma(TM \oplus T^*M) \to \Gamma(TM \oplus T^*M)$ defined by

$$\begin{aligned} [(X_1, \alpha_1), (X_2, \alpha_2)] &:= \left([X_1, X_2] \,, \pounds_{X_1} \alpha_2 - \pounds_{X_2} \alpha_1 + \mathbf{d} \langle \alpha_2, X_1 \rangle \right) \\ &= \left([X_1, X_2] \,, \mathbf{i}_{X_1} \mathbf{d}\alpha_2 - \mathbf{i}_{X_2} \mathbf{d}\alpha_1 + \mathbf{d} \langle \alpha_2, X_1 \rangle \right). \end{aligned}$$

This bracket is the one that Courant [1990] originally developed and it does not necessarily satisfy the Jacobi identity. It was shown by Dorfman [1993] that the integrability condition of the Dirac structure $D \subset TM \oplus T^*M$ given in equation (2.2) can be expressed as

$$[\Gamma(D), \Gamma(D)] \subset \Gamma(D),$$

which is the closedness condition of the Courant bracket (see also Dalsmo and van der Schaft [1998] and Jotz and Ratiu [2008]).

The following Lemma is useful and the proof may be found in Yoshimura and Marsden [2006a] and Courant [1990].

Lemma 2.1. *A subbundle $D \subset TM \oplus T^*M$ is isotropic with respect to $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ if and only if $\langle\!\langle (v, \alpha), (v, \alpha) \rangle\!\rangle = 0, \forall (v, \alpha) \in D$.*

[1]In this paper, all the geometric objects such as manifolds, vector bundles and sections are assumed to be smooth.

Induced Dirac Structures. One of the most important and interesting Dirac structures in mechanics is an *induced Dirac structure* from kinematic constraints, whether holonomic or nonholonomic. Such constraints are generally given by a distribution on a configuration manifold (as to the details, refer to Yoshimura and Marsden [2006a]).

Let Q be a configuration manifold. Let TQ and T^*Q be the tangent and cotangent bundles of Q. Let $\Delta_Q \subset TQ$ be a regular distribution on Q and define a lifted distribution on T^*Q by

$$\Delta_{T^*Q} = (T\pi_Q)^{-1}(\Delta_Q) \subset TT^*Q,$$

where $\pi_Q : T^*Q \to Q$ is the cotangent projection so that its tangent is a map $T\pi_Q : TT^*Q \to TQ$. Let Ω be the canonical two-form on T^*Q. The *induced Dirac structure* D on T^*Q is the subbundle of $TT^*Q \oplus T^*T^*Q$, whose fiber is given for each $(q,p) \in T^*Q$ as

$$\begin{aligned} D(q,p) = \{(v,\alpha) \in T_{(q,p)}T^*Q \times T^*_{(q,p)}T^*Q \mid v \in \Delta_{T^*Q}(q,p), \text{ and} \\ \alpha(w) = \Omega_{\Delta_Q}(q,p)(v,w) \text{ for all } w \in \Delta_{T^*Q}(q,p)\}. \end{aligned}$$

This is, of course, a special instance of the construction (2.1). Notice that the induced Dirac structure can be restated by using the bundle map $\Omega^\flat : TT^*Q \to T^*T^*Q$ as follows:

$$\begin{aligned} D(q,p) = \{(v,\alpha) \in T_{(q,p)}T^*Q \times T^*_{(q,p)}T^*Q \mid v \in \Delta_{T^*Q}(q,p), \text{ and} \\ \alpha - \Omega^\flat(q,p) \cdot v \in \Delta^\circ_{T^*Q}(q,p)\}, \end{aligned}$$

where $\Delta^\circ_{T^*Q}$ is the annihilator of Δ_{T^*Q}.

Remark. If there exists no constraint, namely, the case in which $\Delta_Q = TQ$, the Dirac structure D corresponds to the ***canonical Dirac structure***, which is given by a graph of the bundle map $\Omega^\flat$ associated to Ω.

Local Expressions. Let V be a model space for Q and U be the range in V of a chart on Q. Then, the range of the induced chart on TQ is represented by $U \times V$, while T^*Q is locally given by $U \times V^*$. In this representation, $(q,\dot{q})$ are local coordinates of TQ and (q,p) are local coordinates of T^*Q. Further, TT^*Q is represented by $(U \times V^*) \times (V \times V^*)$, while T^*T^*Q is locally given by $(U \times V^*) \times (V^* \times V)$. In this local representation, $(q,p,\dot{q},\dot{p})$ are the corresponding coordinates of TT^*Q, while (q,p,β,v) are the local coordinates of T^*T^*Q.

Using $\pi_Q : T^*Q \to Q$ locally denoted by $(q,p) \mapsto q$ and $T\pi_Q : (q,p,\dot{q},\dot{p}) \mapsto (q,\dot{q})$, it follows that

$$\Delta_{T^*Q} = \{w = (q,p,\dot{q},\dot{p}) \mid q \in U, \dot{q} \in \Delta(q)\}$$

and the annihilator of Δ_{T^*Q} is locally represented as

$$\Delta^\circ_{T^*Q} = \{\alpha = (q,p,\beta,v) \mid q \in U,\ \beta \in \Delta^\circ(q) \text{ and } v = 0\}.$$

Since we have the local formula $\Omega^\flat(q,p) \cdot w = (q,p,-\dot{p},\dot{q})$, the condition $\alpha - \Omega^\flat(q,p) \cdot w \in \Delta^\circ_{T^*Q}$ reads $\beta + \dot{p} \in \Delta^\circ(q)$ and $v - \dot{q} = 0$. Thus, the induced Dirac structure is locally represented by

$$D(q,p) = \{((\dot{q},\dot{p}),(\beta,v)) \mid \dot{q} \in \Delta(q),\ v = \dot{q}, \quad \beta + \dot{p} \in \Delta^\circ(q)\}, \tag{2.3}$$

where $\Delta^\circ(q) \subset T^*_qQ$ is the polar of $\Delta(q) \subset T_qQ$.

3 Lagrange–Dirac Dynamical Systems

In this section, let us recall the definition of implicit Lagrangian systems or Lagrange-Dirac dynamical systems by following Yoshimura and Marsden [2006a,b].

Lagrange–Dirac Dynamical Systems. Let $L : TQ \to \mathbb{R}$ be a Lagrangian, possibly degenerate. Let D be a Dirac structure on T^*Q induced from a given distribution $\Delta_Q \subset TQ$.

Recall that a *partial vector field* $X : TQ \oplus T^*Q \to TT^*Q$ is defined as a mapping such that $\tau_{T^*Q} \circ X = \mathrm{pr}_{T^*Q}$, where $\mathrm{pr}_{T^*Q} : TQ \oplus T^*Q \to T^*Q$; $(q, v, p) \mapsto (q, p)$ and $\tau_{T^*Q} : TT^*Q \to T^*Q$; $(q, p, \dot{q}, \dot{p}) \mapsto (q, p)$ is the tangent projection.[2] The partial vector field $X : TQ \oplus T^*Q \to TT^*Q$ is a map that assigns to each point $(q, v, p) \in TQ \oplus T^*Q$, a vector in TT^*Q at the point $(q, p) \in T^*Q$; we write X as

$$X(q, v, p) = (q, p, \dot{q}, \dot{p}),$$

where we note that $\dot{q} = dq/dt$ and $\dot{p} = dp/dt$ are understood to be functions of (q, v, p).

Let $E_L : TQ \oplus T^*Q \to \mathbb{R}$ be the *generalized energy* given by

$$E_L(q, v, p) := \langle p, v\rangle - L(q, v).$$

An ***implicit Lagrangian system*** or a ***Lagrange–Dirac dynamical system*** is the triple (E_L, D, X) that satisfies, for each $(q, v, p) \in TQ \oplus T^*Q$ and with $P = \mathbb{F}L(\Delta_Q)$, namely, $(q, p) = (q, \partial L/\partial v)$,

$$(X(q, v, p), \mathbf{d}E_L(q, v, p)|_{T_{(q,p)}P}) \in D(q, p), \tag{3.1}$$

where $\mathbb{F}L : TQ \to T^*Q$ is the Legendre transformation. In the above, $\mathbf{d}E_L(q, v, p) : T_{(q,v,p)}(TQ \oplus T^*Q) \to \mathbb{R}$ is a linear function on $T_{(q,v,p)}(TQ \oplus T^*Q)$ at each point $(q, v, p) \in TQ \oplus T^*Q$ and hence the restriction $\mathbf{d}E_L(q, v, p)|_{T_{(q,p)}P} \cong (-\partial L/\partial q, v)$ may be regarded as a linear function on $T_{(q,p)}P$ such that $(q, p) = (q, \partial L/\partial v) \in P = \mathbb{F}L(\Delta_Q)$.

Local Expressions. It follows from (2.3) that the Lagrange–Dirac dynamical system $(X, \mathbf{d}E_L|_{TP}) \in D$ is locally given by

$$p = \frac{\partial L}{\partial v}, \quad \dot{q} = v \in \Delta_Q(q), \quad \dot{p} - \frac{\partial L}{\partial q} \in \Delta_Q^\circ(q).$$

Remark. If a partial vector field $X(q, v, p) = (q, p, \dot{q}, \dot{p})$ satisfies the condition for a Lagrange–Dirac dynamical system $(X, \mathbf{d}E_L|_{TP}) \in D$, then the Legendre transformation $(q, p) = (q, \partial L/\partial v) \in P = \mathbb{F}L(\Delta_Q)$ is consistent with the equality of base points and the condition itself gives the second-order condition $\dot{q} = v$. In other words, the partial vector field of the Lagrange–Dirac dynamical system is uniquely given on the graph of the Legendre transformation.

For the case in which no kinematic constraint is imposed, i.e., $\Delta_Q = TQ$, we can develop the standard implicit Lagrangian system, which is expressed in local coordinates as

$$p = \frac{\partial L}{\partial v}, \quad \dot{q} = v, \quad \dot{p} = \frac{\partial L}{\partial q},$$

which we shall call ***implicit Euler–Lagrange equations***. Note that the implicit Euler–Lagrange equations include the Euler–Lagrange equations $\dot{p} = \partial L/\partial q$, the Legendre transformation $p = \partial L/\partial v$ as well as the second-order condition $\dot{q} = v$.

[2] Compare this to the definition of a vector field over T^*Q as a mapping $Y : T^*Q \to TT^*Q$ such that $\tau_{T^*Q} \circ Y = \mathrm{Id}_{T^*Q}$

Example: Harmonic Oscillators. We illustrate the implicit Euler–Lagrange equations by the example of a linear harmonic oscillator. In this case, the configuration space is $Q = \mathbb{R}$ representing the position of a particle on the line. The Lagrangian is given by $L = v^2/2 - q^2/2$. We find the generalized energy to be $E_L(q,v,p) = pv - v^2/2 + q^2/2$. The canonical Dirac structure $D(\equiv \mathrm{graph}(\Omega^\flat))$ can be easily constructed by using the canonical symplectic structure Ω on T^*Q. The Lagrange-Dirac system is given by (E_L, D, X), which satisfies

$$((\dot{q},\dot{p}), \mathbf{d}E_L(q,v,p)|_{TP}) \in D(q,p),$$

and it follows $\mathbf{d}E_L(q,v,p)|_{TP} = \Omega^\flat(q,p)\cdot(\dot{q},\dot{p})$. We see $\mathbf{d}E_L(q,v,p)|_{TP} = vdp + qdq$ and $\Omega^\flat(q,p)(\dot{q},\dot{p}) = -\dot{p}dq + \dot{q}dp$. So, the dynamics is equivalent to the following equations:

$$\dot{q} = v, \quad \dot{p} = -q,$$

together with the Legendre transform $p = v$.

Lagrange-Dirac Dynamical Systems with External Forces. Here, we consider the case of a system with external force fields. Let $\mathrm{pr}_Q : TQ \oplus T^*Q \to Q;\ (q,v,p) \mapsto q$ be a natural projection. An external force field $F : TQ \oplus T^*Q \to T^*Q$ induces a horizontal one-form $\tilde{F}$ on $TQ \oplus T^*Q$ such that, for $(q,v,p) \in TQ \oplus T^*Q$,

$$\tilde{F}(q,v,p)\cdot w = \left\langle F(q,v,p), T\mathrm{pr}_Q(w)\right\rangle,$$

where $\tilde{F} = (q,v,p,F(q,v,p),0,0) \in T^*_{(q,v,p)}(TQ \oplus T^*Q)$ and $w = (q,v,p,\delta q,\delta v,\delta p) \in T_{(q,v,p)}(TQ \oplus T^*Q)$.

Given a Lagrangian $L : TQ \to \mathbb{R}$ (possibly degenerate), a ***Lagrange-Dirac dynamical system with an external force field*** is defined by a *quadruple* (E_L, D, X, F), which satisfies, in coordinates $(q,v,p) \in TQ \oplus T^*Q$,

$$(X(q,v,p), (\mathbf{d}E_L - \tilde{F})(q,v,p)|_{TP}) \in D(q,p). \tag{3.2}$$

It follows that the local expression of a Lagrange-Dirac system in equation (3.2) may be given by

$$\dot{q} = v, \quad \dot{p} - \frac{\partial L}{\partial q} - F(q,v,p) \in \Delta_Q^\circ(q), \quad \dot{q} \in \Delta_Q(q) \tag{3.3}$$

and with

$$p = \frac{\partial L}{\partial v}.$$

The curve $(q(t), v(t), p(t))$, $t_1 \leq t \leq t_2$ in $TQ \oplus T^*Q$ that satisfies the condition (3.2) is a solution curve of (E_L, D, X, F).

Power Balance Law. The time derivative of $E_L(q,v,p) = \langle p, v\rangle - L(q,v)$ restricted to the solution curve $(q(t), v(t), p(t))$ reads

$$\frac{d}{dt}E_L(q(t),v(t),p(t)) = \langle F(q(t),v(t),p(t)), \dot{q}(t)\rangle,$$

where we have employed $p(t) = (\partial L/\partial v)(t)$ and $\dot{q}(t) = v(t)$.

Example: The Harmonic Oscillator with Damping. As before, consider the harmonic oscillator in the setting of $Q = \mathbb{R}$, $L = v^2/2 - q^2/2$, $E_L = pv - v^2/2 + q^2/2$ and $D(\equiv \mathrm{graph}(\Omega^\flat))$ and add a damping force $F : TQ \oplus T^*Q \to T^*Q$ defined by $F(q, v, p) = -(rv)dq$, where $r \in \mathbb{R}^+$ is a positive damping constant. Then, $\tilde{F} = (q, v, p, rv, 0, 0)$. The formulas in equation (3.3) give us the equations:

$$\dot{q} = v, \quad \dot{p} + q + rv = 0$$

with the Legendre transformation $p = v$.

The Hamilton-Pontryagin Principle. Lagrange-Dirac dynamical systems can describe general scenarios such as non-holonomic degenerate Lagrangian systems because of a firm basis in variational structures. Given a Lagrangian $L : TQ \to \mathbb{R}$, unconstrained equations of motion are described by Hamilton's principle:

$$\delta \int_{t_1}^{t_2} L(q(t), \dot{q}(t))dt = 0.$$

From the viewpoint of Dirac geometry, we start rather from the Hamilton-Pontryagin principle, which is given by the stationary condition of the action integral on the space of curves $(q(t), v(t), p(t))$, $t \in [t_1, t_2]$ in $TQ \oplus T^*Q$ as

$$\delta \int_{t_1}^{t_2} L(q(t), v(t)) + \langle p(t), \dot{q}(t) - v(t) \rangle \; dt = 0,$$

where $\dot{q}(t) = dq(t)/dt$. The above variational principle can be restated by using the generalized energy $E_L : TQ \oplus T^*Q \to \mathbb{R}$ as

$$\delta \int_{t_1}^{t_2} \langle p(t), \dot{q}(t) \rangle - E_L(q(t), v(t), p(t)) \; dt = 0.$$

Keeping the endpoints $q(t_0)$ and $q(t_1)$ of $q(t)$ fixed, we can obtain the *implicit Euler–Lagrange equations.*

The Lagrange-d'Alembert-Pontryagin Principle. Consider a mechanical system with kinematic constraints that are given by a constraint distribution Δ_Q on Q. Then, the motion of the mechanical system $c : [t_1, t_2] \to Q$ is said to be constrained if $\dot{c}(t) \in \Delta_Q(c(t))$ for all t, $t_1 \le t \le t_2$. Assume that the distribution Δ_Q is not involutive; that is, $[X(q), Y(q)] \notin \Delta_Q(q)$ for any two vector fields X, Y on Q with values in Δ_Q. Further, let L be a (possibly degenerate) Lagrangian on TQ and let $F : TQ \oplus T^*Q \to T^*Q$ be an external force field. The Lagrange-d'Alembert-Pontryagin principle for a curve $(q(t), v(t), p(t))$, $t_1 \le t \le t_2$, in $TQ \oplus T^*Q$ is represented by

$$\begin{aligned} &\delta \int_{t_1}^{t_2} \langle p(t), \dot{q}(t) \rangle - E_L(q(t), v(t), p(t))\, dt + \int_{t_1}^{t_2} \langle F(q(t), v(t), p(t)), \delta q(t) \rangle \; dt \\ &= \delta \int_{t_1}^{t_2} L(q(t), v(t)) + \langle p(t), \dot{q}(t) - v(t) \rangle \; dt + \int_{t_1}^{t_2} \langle F(q(t), v(t), p(t)), \delta q(t) \rangle \; dt \\ &= 0 \end{aligned}$$

for a given variation $\delta q(t) \in \Delta_Q(q(t))$ and with the constraint $v(t) \in \Delta_Q(q(t))$. Keeping the endpoints of $q(t)$ fixed, it follows

$$\int_{t_1}^{t_2} \left\langle \frac{\partial L}{\partial q} - \dot{p}, \delta q \right\rangle + \left\langle \frac{\partial L}{\partial v} - p, \delta v \right\rangle + \langle \delta p, \dot{q} - v \rangle \; dt + \int_{t_1}^{t_2} F(q, v, p)\, \delta q \, dt = 0 \tag{3.4}$$

for a chosen variation $\delta q(t) \in \Delta_Q(q(t))$, for all $\delta v(t)$ and $\delta p(t)$, and with $v(t) \in \Delta_Q(q(t))$.

Proposition 3.1. *The Lagrange-d'Alembert-Pontryagin principle gives the local expression of equations of motion for nonholonomic mechanical systems with external forces such that*

$$\dot{q} = v \in \Delta_Q(q), \quad \dot{p} - \frac{\partial L}{\partial q} - F(q, v, p) \in \Delta_Q^{\circ}(q), \quad p = \frac{\partial L}{\partial v}. \tag{3.5}$$

Proof. From equation (3.4), it reads

$$\int_{t_1}^{t_2} \left\langle \frac{\partial L}{\partial q} - \dot{p} + F(q, v, p), \delta q \right\rangle + \left\langle \frac{\partial L}{\partial v} - p, \delta v \right\rangle + \langle \delta p, \dot{q} - v \rangle \, dt = 0,$$

which is satisfied for a given variation $\delta q(t) \in \Delta_Q(q(t))$, for all $\delta v(t)$ and $\delta p(t)$, and with the constraint $v(t) \in \Delta_Q(q(t))$. Thus, we obtain equation (3.5). ■

Coordinate Representation. Let Q be a finite dimensional smooth manifold whose dimension is n. Choose local coordinates q^i on Q so that Q is locally represented by an open set $U \subset \mathbb{R}^n$. Then, TQ is locally given by $(q^i, v^i) \in U \times \mathbb{R}^n$. Similarly T^*Q may be locally denoted by $(q^i, p_i) \in U \times \mathbb{R}^n$. The constraint set Δ_Q defines a subspace on each fiber of TQ, which we denote by $\Delta_Q(q) \subset \mathbb{R}^n$ at each point $q \in U$. If the dimension of $\Delta_Q(q)$ is $n - m$, then we can choose a basis $e_{m+1}(q), e_{m+2}(q), \ldots, e_n(q)$ of $\Delta_Q(q)$. Recall that the constraint sets can be also represented by the annihilator $\Delta_Q^{\circ}(q)$, which is spanned by m one-forms $\omega^1, \omega^2, \ldots, \omega^m$. It follows that equation (3.5) can be represented, in coordinates, by employing the Lagrange multipliers μ_a, $a = 1, ..., m$, as follows:

$$\begin{aligned}
\begin{pmatrix} \dot{q}^i \\ \dot{p}_i \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -\frac{\partial L}{\partial q^i} - F_i(q, v, p) \\ v^i \end{pmatrix} + \begin{pmatrix} 0 \\ \mu_a \, \omega_i^a(q) \end{pmatrix}, \\
p_i &= \frac{\partial L}{\partial v^i}, \\
0 &= \omega_i^a(q) \, v^i,
\end{aligned}$$

where we employ the local expression $\omega^a = \omega_i^a \, dq^i$.

4 Interconnection of Dirac Structures

There exist various types of the power conserving interconnections between distinct dynamical systems such as two rigid bodies connected by ball-socket joints, mechanical systems connected by transmission gears, transducers between electric circuits and mechanical systems such as D-C motors, fluid and rigid body interactions, etc. All the examples listed above may be represented by Dirac structures. In this section, we will show how distinct Dirac structures can be interconnected via an *interconnection* Dirac structure to yield another Dirac structure of a fully interconnected dynamical system.

The Direct Sum of Dirac Structures. Before going into details on the case of n distinct Dirac structures, we will begin with the case of $n = 2$ to introduce the direct sum of Dirac structures on manifolds. Let $\daleth(M)$ be the set of (almost) Dirac structures over a manifold M.[3] Let M_1 and M_2 be distinct smooth manifolds. To transition from systems on M_1 and M_2 to one on $M = M_1 \times M_2$, we shall define a *direct sum* associated to distinct Dirac structures.

[3] $\daleth$ is called "Daleth", which is the fourth letter of the Semitic alphabet corresponding to "D".

Definition 4.1 (Direct Sum of Dirac Structures)**.** *Let $D_1 \in \daleth(M_1)$ and $D_2 \in \daleth(M_2)$ be given distinct Dirac structures on M_1 and M_2 respectively. Define the* ***Dirac sum*** *of D_1 and D_2 by*

$$\begin{aligned} D_1 \oplus D_2(m_1, m_2) = \{((v_1, v_2), T_m^* \operatorname{pr}_{M_1}(\alpha_1) + T_m^* \operatorname{pr}_{M_2}(\alpha_2)) \in T_m M \times T_m^* M| \\ (v_1, \alpha_1) \in D_1(m_1),\ \ (v_2, \alpha_2) \in D_2(m_2)\}, \forall m = (m_1, m_2) \in M \end{aligned}$$

We now show that $\oplus$ is a mapping from $\daleth(M_1) \times \daleth(M_2) \to \daleth(M)$.

Proposition 4.2. *If $D_1 \in \daleth(M_1), D_2 \in \daleth(M_2)$, then $D_1 \oplus D_2 \in \daleth(M)$.*

Proof. Let $m = (m_1, m_2) \in M$ and let $(v, \alpha) \in D_1 \oplus D_2(m)$. Notice that $\dim((D_1 \oplus D_2)(m)) = \dim(D_1(m_1)) + \dim(D_2(m_2)) = \dim(M_1 \times M_2)$ for $m = (m_1, m_2) \in M$. By definition, $v = (v_1, v_2) \in T_m M$ and $\alpha = T_m^* \operatorname{pr}_{M_1}(\alpha_1) + T_m^* \operatorname{pr}_{M_2}(\alpha_2)$ for some $\alpha_1 \in T_{m_1}^* M_1$ and $\alpha_2 \in T_{m_2}^* M_2$ such that $(v_1, \alpha_1) \in D_1(m_1)$ and $(v_2, \alpha_2) \in D_2(m_2)$. Then, one has, for each $m \in M$,

$$\begin{aligned} \langle\langle (v, \alpha), (v, \alpha) \rangle\rangle &= 2\langle \alpha, v \rangle \\ &= 2\langle T_m^* \operatorname{pr}_{M_1}(\alpha_1) + T_m^* \operatorname{pr}_{M_2}(\alpha_2), v \rangle \\ &= 2\langle \alpha_1, T_m \operatorname{pr}_{M_1}(v) \rangle + 2\langle \alpha_2, T_m \operatorname{pr}_{M_2}(v) \rangle \\ &= 2\langle \alpha_1, v_1 \rangle + 2\langle \alpha_2, v_2 \rangle \\ &= 0, \end{aligned}$$

since $(v_1, \alpha_1) \in D_1(m_1)$ and $(v_2, \alpha_2) \in D_2(m_2)$. Thus, noting that m is arbitrary, one can prove the theorem by Lemma 2.1. ■

The Dirac Sum of Induced Dirac Structures. Let Q_1 and Q_2 be distinct configuration spaces. Let $\Delta_{Q_1} \subset TQ_1$ and $\Delta_{Q_2} \subset TQ_2$ be constraint distributions, and we can define the induced Dirac structures D_1 and D_2, where we assume that $\Delta_{Q_1} \neq \Delta_{Q_2}$ and $\Delta_{Q_1} \cap \Delta_{Q_2} = \emptyset$.

Let $Q = Q_1 \times Q_2$ be an ***extended configuration manifold*** and $\Delta_Q = \Delta_{Q_1} \times \Delta_{Q_2} \subset TQ = TQ_1 \times TQ_2$ be a distribution on Q. As before, define the induced distribution Δ_{T^*Q} on T^*Q by

$$\Delta_{T^*Q} = (T\pi_Q)^{-1}(\Delta_Q) \subset TT^*Q,$$

where $T\pi_Q : TT^*Q \to TQ$ is the tangent map of the cotangent bundle projection $\pi_Q : T^*Q \to Q$, while the annihilator of Δ_{T^*Q} is defined by, for each $(q, p) \in T^*Q$,

$$\Delta^{\circ}_{T^*Q}(q, p) = \{\alpha \in T^*_{(q,p)} T^*Q \mid \langle \alpha, w \rangle = 0 \ \text{ for all } \ w \in \Delta_{T^*Q}(q, p)\}.$$

Proposition 4.3. *Let Ω_i be the canonical symplectic structures on T^*Q_i and $\Omega_i^\flat : TT^*Q_i \to T^*T^*Q_i$ be the associated bundle maps. Let $\operatorname{pr}_{T^*Q_i} : T^*Q_1 \times T^*Q_2 \to T^*Q_i,\ i = 1, 2$ be the natural projections*[4]*. Denote by D_i the Dirac structures on T^*Q_i induced from $\Delta_{Q_i} \subset TQ_i$. Then, the Dirac sum $D_1 \oplus D_2$ can be represented by the induced Dirac structure on T^*Q from $\Delta_Q = \Delta_{Q_1} \times \Delta_{Q_2} \subset TQ$, which is given by, for each $(q, p) \in T^*Q$,*

$$\begin{aligned} D_1 \oplus D_2(q, p) = \{\, (w, \alpha) \in T_{(q,p)} T^*Q \times T^*_{(q,p)} T^*Q \ \mid \ w \in \Delta_{T^*Q}(q, p) \\ \text{and } \ \alpha - (\Omega_1 \oplus \Omega_2)^\flat (q, p) \cdot w \in \Delta^{\circ}_{T^*Q}(q, p) \,\}, \quad (4.1) \end{aligned}$$

where $(\Omega_1 \oplus \Omega_2)^\flat : TT^*Q \to T^*T^*Q$ is the associated bundle map with $\Omega_1 \oplus \Omega_2 = \operatorname{pr}^*_{T^*Q_1} \Omega_1 + \operatorname{pr}^*_{T^*Q_2} \Omega_2$.

[4]Note that we may identify the cotangent bundle $T^*Q_1 \times T^*Q_2$ with $T^*(Q_1 \times Q_2)$.

Proof. Let $(w, \alpha) \in D_1 \oplus D_2(q,p)$ for each $(q,p) \in T^*Q$. Let $\mathrm{pr}_{TT^*Q_i} : TT^*Q_i \oplus T^*T^*Q_i \to TT^*Q_i$, $i = 1,2$. Now, we may decompose (w, α) as $w = (w_1, w_2)$ and $\alpha = T^*_{(q,p)}\,\mathrm{pr}_{T^*Q_1}(\alpha_1) + T^*_{(q,p)}\,\mathrm{pr}_{T^*Q_2}(\alpha_2)$ such that $(w_1, \alpha_1) \in D_1(q_1, p_1)$ and $(w_2, \alpha_2) \in D_2(q_2, p_2)$. Define the distributions $\Delta_{T^*Q_1} = \mathrm{pr}_{TT^*Q_1}(D_1)$ and $\Delta_{T^*Q_2} = \mathrm{pr}_{TT^*Q_2}(D_2)$. Then, It follows that $\alpha_1 - \Omega_1^\flat(q_1, p_1) \cdot w_1 \in \Delta^\circ_{T^*Q_1}(q_1, p_1)$ and $\alpha_2 - \Omega_2^\flat(q_2, p_2) \cdot w_2 \in \Delta^\circ_{T^*Q_2}(q_2, p_2)$, where $w_1 \in \Delta_{T^*Q_1}(q_1, p_1)$ and $w_2 \in \Delta_{T^*Q_2}(q_2, p_2)$. Noting that $(\Omega_1 \oplus \Omega_2)^\flat(q,p) = \Omega_1^\flat(q_1, p_1) \oplus \Omega_2^\flat(q_2, p_2)$ and $w = (w_1, w_2) \in \Delta_{T^*Q_1}(q_1, p_1) \times \Delta_{T^*Q_2}(q_1, p_2)$, one has $\alpha - (\Omega_1 \oplus \Omega_2)^\flat(q,p) \cdot w \in \Delta^\circ_{T^*Q_1}(q_1, p_1) \times \Delta^\circ_{T^*Q_2}(q_2, p_2)$. Additionally, $T\pi_Q(\Delta_{T^*Q_1} \times \Delta_{T^*Q_2}) = \Delta_{Q_1} \times \Delta_{Q_2} = \Delta_Q \subset TQ$. It follows on each fiber $T_{(q,p)}T^*Q$ that $\Delta_{T^*Q_1} \times \Delta_{T^*Q_2} = \Delta_{T^*Q}$ and it is obvious that $\Delta^\circ_{T^*Q} = \Delta^\circ_{T^*Q_1} \oplus \Delta^\circ_{T^*Q_2}$. Putting these all together gives us $w \in \Delta_{T^*Q}(q,p)$ and $\alpha - (\Omega_1 \oplus \Omega_2)^\flat(q,p) \cdot w \in \Delta^\circ_{T^*Q}(q,p)$, for each $(q,p) \in T^*Q$. Since $\Omega_1 \oplus \Omega_2$ is the canonical symplectic form of T^*Q we find $D_1 \oplus D_2$ is the induced Dirac structure from Δ_Q. ■

An Interconnection Dirac Structure. In order to intertwine distinct Dirac structures, we will introduce a special Dirac structure called *interconnecting Dirac structure.* A simple case may be given by a distribution and its annihilator.

Suppose there exists a constraint distribution $\Delta_c \subset TQ$ due to the interconnection, such that

$$\Delta_Q \neq \Delta_c \quad \text{and} \quad \Delta_Q \cap \Delta_c \neq \emptyset,$$

where ω_Q is a one-form on Q associated with the interconnection. The annihilator $\Delta_Q^\circ \subset T^*Q$ is defined by

$$\Delta_c^\circ(q) = \{ f \in T_q^*Q \mid \langle f, v \rangle = 0 \mid v \in \Delta_c(q) \}.$$

We can define the lifted distribution Δ_c on T^*Q by

$$\Delta_{\mathrm{int}} = (T\pi_Q)^{-1}(\Delta_c) \subset TT^*Q,$$

where $T\pi_Q : TT^*Q \to TQ$ is the tangent map of the cotangent bundle projection $\pi_Q : T^*Q \to Q$, while the annihilator of Δ_{int} is given by, for each $(q,p) \in T^*Q$,

$$\Delta^\circ_{\mathrm{int}}(q,p) = \{ \alpha \in T^*_{(q,p)}T^*Q \mid \langle \alpha, w \rangle = 0 \ \text{ for all } \ w \in \Delta_{\mathrm{int}}(q,p) \}.$$

Then, the ***interconnection Dirac structure*** associated to the distribution Δ_{int} on T^*Q can be defined by, for each $(q,p) \in T^*Q$,

$$\begin{aligned} D_{\mathrm{int}}(q,p) = \{ \, (w, \alpha) \in T_{(q,p)}T^*Q \times T^*_{(q,p)}T^*Q \ \mid \ & w \in \Delta_{\mathrm{int}}(q,p) \\ & \text{and} \ \ \alpha \in \Delta^\circ_{\mathrm{int}}(q,p) \, \}. \end{aligned} \tag{4.2}$$

Example: Two Particles Moving with Contact. Consider two masses moving with contact in a vector space $V = \mathbb{R}^2$, whose velocities are given by $(v_1, v_2) \in V$. Since the two particles are in contact and their velocities are common, it follows

$$(v_1, v_2) \in \Delta_c \subset V,$$

where $\Delta_c = \{ (v_1, v_2) \mid v_1 = v_2 \}$ is a constraint subspace of V. This constraint is enforced through the associated constraint forces $(F_1, F_2) \in V^*$ at the contact point, where F_i is the force associated with the i-th particle. This implies Newton's third law and it follows

$$(F_1, F_2) \in \Delta_c^\circ \subset V^*,$$

where $\Delta_c^\circ = \{ (F_1, F_2) \mid F_1 = -F_2 \}$ is the annihilator of Δ_c. Note that we can define a Dirac structure on V by $D_V = \Delta_c \times \Delta_c^\circ$. Then, it follows that $(v_1, F_1), (v_2, F_2) \in D_V$.

Remark. We can consider a more general interconnection Dirac structure than stated in the above, such as those given by gyrators, transformers and so on, which are expressed by two-forms or Poisson structures. In this paper, we will mainly focus on the simple case of the interconnection Dirac structure D_{int} induced from a given distribution Δ_c

Strategy for the Interconnection of Distinct Dirac Structures. Before going into details, let us outline our strategy for interconnecting Dirac structures. Let D_1 and D_2 be two distinct Dirac structures on distinct manifolds M_1 and M_2. We will interconnect D_1 and D_2 by introducing an *interconnection* Dirac structure D_{int} on $M = M_1 \times M_2$, by which constraints due to the interconnection are imposed. To do this, we first make the direct sum of Dirac structures, namely $D_1 \oplus D_2$, which is a single Dirac structure on M. Then, by introducing the *bowtie product* $\bowtie$ of Dirac structures on M, we define an interconnected Dirac structure as $(D_1 \oplus D_2) \bowtie D_{\mathrm{int}}$. The bowtie product will be shown to make $(\urcorner(M), \bowtie)$ a commutative category with identity element $TM \oplus \{0\}$. Under the condition that the fiber dimension of $\mathrm{pr}_{TM}(D_1) \cap \mathrm{pr}_{TM}(D_2)$ is constant, it will be shown that $(D_1 \oplus D_2) \bowtie D_{\mathrm{int}}$ is indeed a Dirac structure on M.

The Bowtie Product. Let $P = M \times M$ and let $\iota : \bar{M} \hookrightarrow P$ be a submanifold given as $\bar{M} := \{(m, n) \in P \mid m = n\}$. Given Dirac structures D_a, D_b on M, define the direct product of Dirac structures D_a and D_b by $D_a \oplus D_b$ on P. Recall from Courant [1990] that one may pull back the Dirac structure $D_a \oplus D_b \subset TP \oplus T^*P$ on P to $\bar{M}$ as

$$\iota^*(D_a \oplus D_b) := \frac{(D_a \oplus D_b) \cap (T\bar{M} \oplus T^*P|_{\bar{M}})}{(D_a \oplus D_b) \cap (\{0\} \oplus T\bar{M}^\circ)},$$

which is a Lagrangian submanifold of $T\bar{M} \oplus T^*\bar{M}$. Assume that $\iota^*(D_a \oplus D_b)$ is a smooth sub-bundle by either of the following conditions:

- $(D_a \times D_b) \cap (T\bar{M} \oplus T^*P)$ has constant dimension,
- $(D_a \times D_b) \cap (\{0\} \oplus T\bar{M}^\circ)$ has constant dimension.

Corollary 4.4. *The subbundle $\iota^*(D_a \oplus D_b)$ is a Dirac structure on $\bar{M}$.*

Let $\varsigma : M \to \bar{M}$ be the map $m \mapsto (m, m)$ (this is a diffeomorphism between M and $\bar{M}$). Then $d := \iota \circ \varsigma : M \to P$ given by $m \to (m, m)$ is called the *diagonal embedding*. Given Dirac structures $D_a, D_b \in \urcorner(M)$, we may define the Dirac structure $D_a \oplus D_b \in \urcorner(M \times M)$. Noting $\iota^*(D_a \oplus D_b) \in \urcorner(\bar{M})$, it follows that

$$d^*(D_a \oplus D_b) = \varsigma^*[\iota^*(D_a \oplus D_b)] \in \urcorner(M).$$

Definition 4.5. *Let D_a and D_b be Dirac structures over M. Then, we define a tensor product called the* ***bowtie product*** *of D_a and D_b by*

$$\begin{aligned} D_a \bowtie D_b = \{(v, \alpha) \in TM \oplus T^*M \mid \exists \beta \in T^*M \\ \textit{such that } (v, \alpha + \beta) \in D_a, (v, -\beta) \in D_b\}, \end{aligned} \tag{4.3}$$

or alternatively given by:

$$D_a \bowtie D_b = d^*(D_a \oplus D_b). \tag{4.4}$$

Theorem 4.6. *Under the assumption that $(D_a \oplus D_b) \cap (\{0\} \oplus T\bar{M}^\circ)$ has constant dimension, $D_a \bowtie D_b$ is a Dirac structure on M.*

Remark. In this paper, we focus on how to use the bowtie product $\bowtie$ to make a Dirac structure for an interconnected Lagrange–Dirac dynamical system. While the bowtie product was initially proposed in Yoshimura, Jacobs and Marsden [2010] for this purpose using formula (4.3), it was revealed that an equivalent *tensor product* (denoted by $\boxtimes$) was also developed in the context of generalized complex geometry (see Gualtieri [2007] and Alekseev, Bursztyn and Meinrenken [2009]) using formula (4.4).[5] We will make explicit the equivalence of $\bowtie$ and $\boxtimes$ (i.e. (4.3) and (4.4)) in the Appendix.

Remark. For standard Lagrange-Dirac mechanics, we usually employ $M = T^*Q$, D_a is the canonical Dirac structure on M, and $D_b = \Delta \oplus \Delta^\circ$ for some distribution $\Delta \subset TM$ is an interconnection Dirac structure. Though we will not explore this in detail, one of the particularly interesting interconnection Dirac structures is the case in which D_b can be developed by Lagrangians that are linear in velocities of the form $L = \langle \alpha, u \rangle$, where $u \in TQ$ and α is a one-form on Q causing magnetic terms to appear.

In order to show interconnection of Dirac structures is associative, we will use a special restricted two-form Ω_Δ induced by a Dirac structure D on M with a distribution $\Delta = \mathrm{pr}_{TM}(D)$, where $\mathrm{pr}_{TM} : TM \oplus T^*M \to TM$; $(v, \alpha) \mapsto v$. This two-form will be employed in the proofs for theorems 4.9 and 4.11.

Lemma 4.7. *Let $D \in \daleth(M)$ and set $\Delta = \mathrm{pr}_{TM}(D)$. On each fiber of $T_xM \times T_x^*M$ at $x \in M$ there exists a bilinear anti-symmetric map $\Omega_\Delta(x) : \Delta(x) \times \Delta(x) \to \mathbb{R}$ defined as*

$$\Omega_\Delta(x)(v_1, v_2) = \langle \alpha_1, v_2 \rangle \text{ such that } (v_1, \alpha_1) \in D(x). \tag{4.5}$$

This two-form was initially introduced by Courant [1990] for linear Dirac structures (see also Dufour and Wade [2004]). We can easily generalize it to the case of general manifolds since Ω_Δ is defined fiber-wise, and hence we have that $D(x)$ is a linear Dirac structures on each fiber $T_xM \times T_x^*M$ for all $x \in M$. Here, we present a proof for completeness.

Proof. We desire to show that equation (4.5) defines a unique bi-linear anti-symmetric map Ω_Δ.

First we prove linearity. Let $(v_1, \alpha_1), (v_2, \alpha_2) \in D(x)$ and let $c_1, c_2 \in \mathbb{R}$. Since $D(x)$ is a linear space at each $x \in M$ we have $(c_1v_1 + c_2v_2, c_1\alpha_1 + c_2\alpha_2) \in D(x)$. By definition of Ω_Δ, it follows, for all $w \in \Delta(x)$,

$$\begin{aligned}\Omega_\Delta(x)(c_1v_1 + c_2v_2, w) &= \langle c_1\alpha_1 + c_2\alpha_2, w \rangle \\ &= c_1\langle \alpha_1, w \rangle + c_2\langle \alpha_2, w \rangle \\ &= c_1\Omega_\Delta(x)(v_1, w) + c_2\Omega_\Delta(x)(v_2, w).\end{aligned}$$

Thus, Ω_Δ is linear in the first argument. Second, in order to prove that Ω_Δ is anti-symmetric, take any $(v_1, \alpha_1), (v_2, \alpha_2) \in D(x)$. Since D is isotropic, we find

$$\langle\langle (v_1, \alpha_1), (v_2, \alpha_2) \rangle\rangle = \langle \alpha_1, v_2 \rangle + \langle \alpha_2, v_1 \rangle = 0$$

and we compute

$$\Omega_\Delta(x)(v_1, v_2) = \langle \alpha_1, v_2 \rangle = -\langle \alpha_2, v_1 \rangle = -\Omega_\Delta(x)(v_2, v_1).$$

Thus, we have shown anti-symmetry from which bi-linearity follows. ■

[5]We appreciate Henrique Bursztyn for pointing out this fact in private communication.

Definition 4.8. *Given a Dirac structure $D \in \daleth(M)$, we call the two-form Ω_Δ of Lemma 4.7 the* ***Dirac two-form****.*

Recall from equation (2.1) that, for each $x \in M$, $D(x)$ may be given by

$$D(x) = \{(v, \alpha) \in T_xM \times T_x^*M \ \mid v \in \Delta(x), \text{ and } \alpha(w) = \Omega_\Delta(x)(v, w) \text{ for all } w \in \Delta(x)\},$$

which may be also stated by

$$D(x) = \{(v, \alpha) \in T_xM \times T_x^*M \mid v \in \Delta(x), \text{ and } \alpha - \Omega^\flat(x) \cdot v \in \Delta^\circ(x)\},$$

where $\Omega^\flat(x) : T_xM \to T_x^*M$ is the skew-symmetric bundle map that is a natural extension of the skew-symmetric map $\Omega^\flat_\Delta(x) : \Delta(x) \subset T_xM \to \Delta^*(x) = T_x^*M/\Delta^\circ(x) \subset T_x^*M$, which is defined by $\langle \Omega^\flat_\Delta(x)(v_x), w_x\rangle = \Omega_\Delta(v_x, w_x)$ on $\Delta(x)$.

Theorem 4.9. *Let D_a and $D_b \in \daleth(M)$. Let $\Delta_a = \mathrm{pr}_{TM}(D_a), \Delta_b = \mathrm{pr}_{TM}(D_b)$ and let Ω_a and Ω_b be the Dirac two-forms for D_a and D_b respectively. If $\Delta_a \cap \Delta_b$ has constant rank, then $D_a \bowtie D_b$ is a Dirac structure with distribution $\mathrm{pr}_{TM}(D_a \bowtie D_b) = \Delta_a \cap \Delta_b$ and with the Dirac two-form $(\Omega_a + \Omega_b)|_{\Delta_a \cap \Delta_b}$.*

Proof. Let $(v, \alpha) \in D_a \bowtie D_b(x)$ for each $x \in M$. Then, there exists $\beta \in T_x^*M$ such that $(v, \alpha + \beta) \in D_a(x), (v, -\beta) \in D_b(x)$ for each $x \in M$. This is equivalent to

$$\Omega^\flat_a(x) \cdot v - \alpha - \beta \in \Delta^\circ_a(x) \text{ and } \Omega^\flat_b(x) \cdot v + \beta \in \Delta^\circ_b(x), \text{ for each } x \in M,$$

where $v \in \Delta_a(x)$ and $v \in \Delta_b(x)$. This means $(\Omega^\flat_a + \Omega^\flat_b)(x) \cdot v - \alpha \in \Delta^\circ_a(x) + \Delta^\circ_b(x)$ and $v \in \Delta_a \cap \Delta_b(x)$. But $\Delta^\circ_a(x) + \Delta^\circ_b(x) = (\Delta_a \cap \Delta_b)^\circ(x)$. Therefore, upon setting $\Omega_c = \Omega_a + \Omega_b$ and $\Delta_c = \Delta_a \cap \Delta_b$, we can write $\Omega^\flat_c(x) \cdot v - \alpha \in \Delta^\circ_c(x)$ and $v \in \Delta_c(x)$; namely, $(v, \alpha) \in D_c(x)$, where D_c is a Dirac structure with Δ_c and Ω_c. Then, it follows that $D_a \bowtie D_b \subset D_c$. Equality follows from noting that both $D_a \bowtie D_b(x)$ and $D_c(x)$ are subspaces of $T_xM \times T_x^*M$ with the same dimension. ■

Corollary 4.10. *If $\Omega_b = 0$, then it follows that $D_b = \Delta_b \oplus \Delta^\circ_b$ and also that $D_c = D_a \bowtie D_b$ is induced by the constraint $\Delta_a \cap \Delta_b$ and the two-form Ω_a.*

Theorem 4.11. *Let $D_a, D_b, D_c \in \daleth(M)$ with distributions $\Delta_a = \mathrm{pr}_{TM}(D_a)$, $\Delta_b = \mathrm{pr}_{TM}(D_b)$, and $\Delta_c = \mathrm{pr}_{TM}(D_c)$. Assume that $\Delta_a \cap \Delta_b$, $\Delta_b \cap \Delta_c$ and $\Delta_c \cap \Delta_a$ have constant ranks. Then the bowtie product $\bowtie$ is associative and commutative; namely we have*

$$(D_a \bowtie D_b) \bowtie D_c = D_a \bowtie (D_b \bowtie D_c)$$

and

$$D_a \bowtie D_b = D_b \bowtie D_a.$$

Proof. First we prove commutativity. Recall that any Dirac structure may be constructed by its associated constraint distribution $\Delta = \mathrm{pr}_{TM}(D)$ and the Dirac two-form Ω_Δ. Let Ω_a, Ω_b,and Ω_c be the Dirac two-forms corresponding to D_a, D_b, and D_c respectively. Then we find by Theorem 4.9 that $D_a \bowtie D_b$ is defined by the distribution $\Delta_{ab} = \Delta_a \cap \Delta_b$ and the Dirac two-form $\Omega_{\Delta_{ab}} = (\Omega_{\Delta_a} + \Omega_{\Delta_b})|_{\Delta_{ab}}$. By commutativity of $+$ and $\cap$, we find the same distribution and the two-form for $D_b \bowtie D_a$, we have $D_a \bowtie D_b = D_b \bowtie D_a$.

Now we prove associativity. Let $\Delta_{(ab)b} = \mathrm{pr}_{TM}((D_a \bowtie D_b) \bowtie D_c)$ and $\Delta_{a(bc)} = \mathrm{pr}_{TM}(D_a \bowtie (D_b \bowtie D_c))$ and it follows

$$\Delta_{(ab)c} = (\Delta_a \cap \Delta_b) \cap \Delta_c = \Delta_a \cap (\Delta_b \cap \Delta_c) = \Delta_{a(bc)}.$$

If $\Omega_{\Delta_{(ab)c}}$ and $\Omega_{\Delta_{a(bc)}}$ are respectively the Dirac two-forms for $(D_a \bowtie D_b) \bowtie D_c$ and $D_a \bowtie (D_b \bowtie D_c)$, we find

$$\begin{aligned}\Omega_{\Delta_{(ab)c}} &= [(\Omega_{\Delta_a} + \Omega_{\Delta_b})|_{\Delta_{ab}} + \Omega_{\Delta_c}]|_{\Delta_{(ab)c}} \\ &= (\Omega_{\Delta_a} + \Omega_{\Delta_b} + \Omega_{\Delta_c})|_{\Delta_{(ab)c}} \\ &= (\Omega_{\Delta_a} + \Omega_{\Delta_b} + \Omega_{\Delta_c})|_{\Delta_{a(bc)}} \\ &= \Omega_{\Delta_{a(bc)}}.\end{aligned}$$

Thus, we obtain

$$(D_a \bowtie D_b) \bowtie D_c = D_a \bowtie (D_b \bowtie D_c).$$

■

Remark. We have shown that $\bowtie$ acts on pairs of Dirac structures with clean intersections to give a new Dirac structure and also that that it is an associative and commutative product. It is easy to verify from equation (4.3) that the Dirac structure $D_e = TM \oplus \{0\}$ satisfies the property of the identity element as $D_e \bowtie D = D = D \bowtie D_e$ for every $D \in \daleth(M)$. Thus we find that $(\daleth(M), \bowtie)$ is a commutative category (it lacks totality and an inverse element, so it is neither a monoid nor a groupoid).

Interconnection of Two Distinct Induced Dirac Structures. Let D_1 and D_2 be Dirac structures on distinct manifolds Q_1 and Q_2 induced from constraint distributions $\Delta_{Q_1} \subset TQ_1$ and $\Delta_{Q_2} \subset TQ_2$ as before. Let Δ_c be a given distribution on $Q = Q_1 \times Q_2$ due to the interconnection of D_1 and D_2. We have mathematical ingredients for *interconnecting* the induced Dirac structures D_1 and D_2 through the interconnection Dirac structure $D_{\rm int}$ on Q. Recall that an interconnection Dirac structure is given by $D_{\rm int} = \Delta_{\rm int} \oplus \Delta_{\rm int}^{\circ}$, where $\Delta_{\rm int} = (T\pi_Q)^{-1}(\Delta_c) \subset TT^*Q$ is the constraint distribution associated to the interconnection. Recall that $\Delta_{T^*Q_1} = (T\pi_{Q_1})^{-1}(\Delta_{Q_1}) \subset TT^*Q_1$ and $\Delta_2 = (T\pi_{Q_2})^{-1}(\Delta_{Q_2}) \subset TT^*Q_2$. It follows from Theorem 4.9 that $D = (D_1 \oplus D_2) \bowtie D_{\rm int}$ may be constructed from the constraint $\Delta = (\Delta_1 \times \Delta_2) \cap \Delta_{\rm int}$ and $\Omega = \Omega_1 \oplus \Omega_2$, where Ω_1 and Ω_2 are respectively the canonical two-forms on T^*Q_1 and T^*Q_2.

Proposition 4.12. *Assuming Δ has constant rank, the* ***interconnection of two distinct induced Dirac structures*** *D_1 and D_2 through $D_{\rm int} = \Delta_{\rm int} \oplus \Delta_{\rm int}^{\circ}$, namely,*

$$D = (D_1 \oplus D_2) \bowtie D_{\rm int}$$

is represented by,

$$\begin{aligned}D(q,p) = \{\, (w, \alpha) \in T_{(q,p)}T^*Q \times T^*_{(q,p)}T^*Q \mid \\ w \in \Delta(q,p) \;\; \textit{and} \;\; \alpha - \Omega^{\flat}(q,p) \cdot w \in \Delta^{\circ}(q,p) \,\}.\end{aligned}$$

This may be viewed as a Corollary to Theorem 4.9 and Proposition 4.1.

Interconnection of n Distinct Dirac Structures. Let us consider the interconnection of n distinct Dirac structures $D_1, D_2, \ldots, D_n$ on distinct manifolds $M_1, M_2, \ldots, M_n$, in which $n > 2$. Recall that the Dirac sum $\oplus$ is associative as $(D_a \oplus D_b) \oplus D_c = D_a \oplus (D_b \oplus D_c)$, so that we may define the iterated Direct sum by

$$\bigoplus_{i=1}^{N} D_i = D_1 \oplus D_2 \oplus \cdots \oplus D_n.$$

By choosing the appropriate interconnection Dirac structure

$$D_{\mathrm{int}} \in \daleth(M_1 \times \cdots \times M_n)$$

and assuming that $\mathrm{rank}(\mathrm{pr}_{TM}(\oplus D_i) \cap \mathrm{pr}_{TM}(D_{\mathrm{int}}))$ is constant, we can simply develop the interconnected Dirac structure for N distinct Dirac structures as

$$D = \left(\bigoplus_{i=1}^{n} D_i \right) \bowtie D_{\mathrm{int}}. \tag{4.6}$$

The condition that $\mathrm{rank}\,(\mathrm{pr}_{TM}(\oplus D_i) \cap \mathrm{pr}_{TM}(D_{\mathrm{int}}))$ is constant is equivalent to the assumption that $\mathrm{rank}(\Delta)$ is constant, which can be shown by resorting to local coordinate expressions.

A Link Between Composition and Interconnection of Dirac Structures. The notion of *composition* of Dirac structures was introduced in Cervera, van der Schaft, and Banos [2007] in the context of port-Hamiltonian systems. Let V_1, V_2 and V_s be vector spaces. Let D_1 be a linear Dirac structure on $V_1 \oplus V_s$ and D_2 be a linear Dirac stucture on $V_s \oplus V_2$. The *composition* of D_1 and D_2 is given by

$$\begin{aligned} D_1 || D_2 = \{ & (v_1, v_2, \alpha_1, \alpha_2) \in (V_1 \times V_2) \oplus (V_1^* \times V_2^*) \mid \\ & \exists (v_s, \alpha_s) \in V_s \oplus V_s^*, \text{ such that } (v_1, v_s, \alpha_1, \alpha_s) \in D_1, (-v_s, v_2, \alpha_s, \alpha_s) \in D_2 \}, \end{aligned}$$

where V_1^*, V_2^* and V_s^* denote the dual space of V_1, V_2 and V_s. It was also shown that the set $D_1||D_2$ is itself a Dirac structure on $V_1 \times V_2$. In this section, we shall see the link between the composition and the interconnection of Dirac structures.

The composition developed in Cervera, van der Schaft, and Banos [2007] is constructed on vector-spaces. So, we also focus on the vector space case in this paper. Let $V = V_1 \times V_s \times V_s \times V_2$ and $\bar{V} = V_1 \times V_2$. Before going into details, we establish an important fact on the projection from V to $\bar{V}$. Set $\Psi : V \to \bar{V}$ to be the projection $(v_1, v_s, v_s', v_2) \mapsto (v_1, v_2)$. The mapping $\Psi^* : \bar{V}^* \to V^*$ dual to Ψ is given by $\Psi^*(\alpha_1, \alpha_2) = (\alpha_1, 0, 0, \alpha_2) \in V^*$.

Next, we recall the push-forward map associated to a Dirac structure. Let U and W be vector spaces. Let $\varphi : U \to W$ be a linear map and let D_U be a linear Dirac structure on U. The push-forward of D_U to W by φ is the set

$$D_W = \{ (\varphi(v), \alpha) \in W \oplus W^* \mid v \in U,\ \alpha \in W^*,\ (v, \varphi^*(\alpha)) \in D_U \}.$$

It is worth noting that the push-forward of a Dirac structure $\varphi_* D_U$ is itself a Dirac structure D_W; namely, we write it as

$$\varphi_* D_U = D_W.$$

See, for instance, Bursztyn and Radko [2003] and Yoshimura and Marsden [2007b].

We are now ready to provide the link between the bowtie product and the composition of Dirac structures.

Proposition 4.13. *Let $\Delta_{\mathrm{int}} = \{(v_1, v_s, -v_s, v_2) \in V\}$. Set $D_{\mathrm{int}} = \Delta_{\mathrm{int}} \oplus \Delta_{\mathrm{int}}^{\circ}$. Set $\Psi : V \to \bar{V}$ to be the projection $(v_1, v_s, v_s', v_2) \mapsto (v_1, v_2)$. For linear Dirac structures D_1 on $V_1 \times V_s$ and D_2 on $V_s \times V_2$, set*

$$D_{\bowtie} = (D_1 \oplus D_2) \bowtie D_{\mathrm{int}}$$

and set also

$$D_{||} = D_1 || D_2.$$

Then, one has

$$D_{||} = \Psi_* D_{\bowtie}.$$

Proof. First, we observe that $\Delta_{\mathrm{int}}^{\circ} = \{(0, \alpha_s, \alpha_s, 0) \in V^*\}$. We also observe

$$\Psi_* D_{\bowtie} = \{(\Psi(v_1, v_s, v_s', v_2), \alpha_1, \alpha_2) \mid (v_1, v_s, v_s', v_2, \Psi^*(\alpha_1, \alpha_2)) \in D_{\bowtie}\}$$

by definition of the push-forward map. Using the facts that $\Psi(v_1, v_s, v_s', v_2) = (v_1, v_2)$ and $\Psi^*(\alpha_1, \alpha_2) = (\alpha_1, 0, 0, \alpha_2) \in V^*$,

$$\Psi_* D_{\bowtie} = \{(v_1, v_2, \alpha_1, \alpha_2) \mid \exists v_s, v_s' \in V_s \text{ such that } (v_1, v_s, v_s', v_2, \alpha_1, 0, 0, \alpha_2) \in D_{\bowtie}\}.$$

Further, it follows from the definition of the bowtie product that

$$\begin{aligned}\Psi_* D_{\bowtie} = \{(v_1, v_2, \alpha_1, \alpha_2) \mid \exists v_s, v_s' \in V_s \text{ and } \exists \beta \in V^* \text{ such that}\\ (v_1, v_s, v_s', v_2, \alpha_1 + \beta_1, \beta_s, \beta_s', \alpha_2 + \beta_2) \in D_1 \oplus D_2,\\ (v_1, v_s, v_s', v_2, -\beta_1, -\beta_s, -\beta_s', -\beta_2) \in D_{\mathrm{int}}\}.\end{aligned}$$

Utilizing the fact that $(v_1, v_s, v_s', v_2, -\beta_1, -\beta_s, -\beta_s', -\beta_2) \in D_{\mathrm{int}}$ if and only if $v_s = -v_s'$ and $\beta_s = \beta_s', \beta_1 = 0, \beta_2 = 0$, we may restate the above as

$$\begin{aligned}\Psi_* D_{\bowtie} = \{(v_1, v_2, \alpha_1, \alpha_2) \mid \exists v_s \in V_s, \alpha_s \in V_s^* \text{ such that}\\ (v_1, v_s, -v_s, v_2, \alpha_1, \alpha_s, \alpha_s, \alpha_2) \in D_1 \oplus D_2\}.\end{aligned}$$

Finally, we have

$$\begin{aligned}\Psi_* D_{\bowtie} = \{(v_1, v_2, \alpha_1, \alpha_2) \mid \exists v_s \in V_s, \alpha_s \in V_s^* \text{ such that}\\ (v_1, v_s, \alpha_1, \alpha_s) \in D_1, \ \ (-v_s, v_2, \alpha_s, \alpha_2) \in D_2\},\end{aligned}$$

but this is nothing but the composition of Dirac structures, namely, $D_{||}$. ■

5 Interconnected Lagrange-Dirac Dynamical Systems

Given distinct mechanical systems, one can develop dynamics of each mechanical system in the context of both the Lagrange-d'Alembert-Pontryagin variational structures and the induced Dirac structures over separate configuration manifolds. In this section, we will show how dynamics of an interconnected systems can be formulated in the context of the induced Dirac structures as well as in the context of the variational structures.

Variational Structures for Interconnected Mechanical Systems. Let us introduce an interconnection of the Lagrange-d'Alembert-Pontryagin principles for distinct mechanical systems.

Definition 5.1. *Let Q_i be distinct configuration manifolds corresponding to distinct mechanical systems $i = 1, ..., n$. Let $E_{L_i} : TQ_i \oplus T^*Q_i \to \mathbb{R}$ be the generalized energies associated to given Lagrangians (possibly degenerate) L_i on TQ_i and $F_i : TQ_i \oplus T^*Q_i \to T^*Q_i$ be external force fields. Let Δ_{Q_i} be a distribution on Q_i for $i = 1, ..., n$.*

*The **Lagrange-d'Alembert-Pontryagin principles** for curves $(q_i(t), v_i(t), p_i(t))$, $t_1 \leq t \leq t_2$ in $TQ_i \oplus T^*Q_i$ is denoted by*

$$\begin{aligned}&\delta \int_{t_1}^{t_2} \langle p_i(t), \dot{q}_i(t)\rangle - E_{L_i}(q_i(t), v_i(t), p_i(t))\, dt + \int_{t_1}^{t_2} \langle F_i(q_i(t), v_i(t), p_i(t)), \delta q_i(t)\rangle\, dt\\ &= \delta \int_{t_1}^{t_2} L_i(q_i(t), v_i(t)) + \langle p_i(t), \dot{q}_i(t) - v_i(t)\rangle\ dt + \int_{t_1}^{t_2} \langle F_i(q_i(t), v_i(t), p_i(t)), \delta q_i(t)\rangle\, dt\\ &= 0, \qquad (5.1)\end{aligned}$$

for $\delta q_i(t) \in \Delta_{Q_i}(q_i(t))$ and with $\dot{q}_i(t) \in \Delta_{Q_i}(q_i(t))$.

*The **interconnection of distinct Lagrange-d'Alembert-Pontryagin principles** may be given by imposing some distribution Δ_c on the extended configuration manifold $Q = Q_1 \times \cdots \times Q_n$ such that*

$$(\dot{q}_1, ..., \dot{q}_n) \in \Delta_c(q_1, ..., q_n) \quad \text{and} \quad F_1(q_1, v_1, p_1) \oplus \cdots \oplus F_n(q_n, v_n, p_n) \in \Delta_c^{\circ}(q_1, ..., q_n). \quad (5.2)$$

Proposition 5.2. *It follows from equations (5.1) and (5.2) that, by using variations $\delta q_i(t)$ vanishing at the endpoints, one can obtain*

$$\dot{q}_i = v_i \in \Delta_{Q_i}(q_i), \quad \dot{p}_i - \frac{\partial L_i}{\partial q_i} - F_i \in \Delta_{Q_i}^{\circ}(q_i), \quad p_i = \frac{\partial L_i}{\partial v_i}, \quad i = 1, ..., n. \quad (5.3)$$

Proof. By direct computations and integration by parts, it follows that

$$\begin{aligned} &\delta \int_{t_1}^{t_2} L_1(q_i, v_i) + \langle p_i, \dot{q}_i - v_i \rangle \, dt + \int_{t_1}^{t_2} \langle F_i, \delta q_i \rangle \, dt \\ &= \delta \int_{t_1}^{t_2} \left\langle \frac{\partial L_i}{\partial q_i} - \dot{p}_i + F_i, \delta q_i \right\rangle + \langle \delta p_i, \dot{q}_i - v_i \rangle + \left\langle p_i - \frac{\partial L_i}{\partial v_i}, \delta q_i \right\rangle \, dt \\ &= 0 \end{aligned}$$

for $\delta q_i \in \Delta_{Q_i}(q_i)$ and with $\dot{q}_i \in \Delta_{Q_i}(q_i)$. Imposing the interconnection condition (5.2), we obtain ***implicit Lagrange-d'Alembert equations for the interconnected system***. ■

An Alternative Lagrange-d'Alembert-Pontryagin Principle. Here, we show that there exists an alternative Lagrange-d'Alembert-Pontryagin principle. Let $Q = Q_1 \times \cdots \times Q_n$ and let $q = (q_1, ..., q_n) \in Q$, $(q, v) = (q_1, ..., q_n, v_1, ..., v_n) \in TQ \cong TQ_1 \times \cdots \times TQ_n$ and $(q, p) = (q_1, ..., q_n, p_1, ..., p_n) \in T^*Q \cong T^*Q_1 \times \cdots \times T^*Q_n$.

Define the generalized energy for the interconnected system by $E_L = E_{L_1} + \cdots + E_{L_n} : TQ \oplus T^*Q \to \mathbb{R}$, where $L = L_1 + \cdots + L_n : TQ \to \mathbb{R}$ is the Lagrangian for the interconnected system. Set the constraint distribution $\Delta = (\Delta_{Q_1} \times \cdots \times \Delta_{Q_n}) \cap \Delta_c \subset TQ$.

The Lagrange-d'Alembert-Pontryagin principle for a curve $(q(t), v(t), p(t)),\ t_1 \leq t \leq t_2$ in $TQ \oplus T^*Q$ may be given by

$$\begin{aligned} &\delta \int_{t_1}^{t_2} \langle p(t), \dot{q}(t) \rangle - E_L(q(t), v(t), p(t)) \, dt \\ &= \delta \int_{t_1}^{t_2} L(q(t), v(t)) + \langle p(t), \dot{q}(t) - v(t) \rangle \, dt = 0, \end{aligned} \quad (5.4)$$

for the chosen variation $\delta q(t) \in \Delta(q(t)) \subset T_{q(t)}Q$, for all $\delta v(t)$ and $\delta p(t)$, and with $\dot{q}(t) \in \Delta(q(t)) \subset T_{q(t)}Q$.

Keeping the endpoints of $q(t)$ fixed, it follows that one can simply derive the Lagrange-d'Alembert-Pontryagin equations as follows:

$$\dot{q} = v \in \Delta(q), \quad \dot{p} - \frac{\partial L}{\partial q} \in \Delta^{\circ}(q), \quad p = \frac{\partial L}{\partial v}.$$

Proposition 5.3. *The Lagrange-d'Alembert-Pontryagin principles in (5.1) for curves $(q_i(t), v_i(t), p_i(t))$, $t_1 \leq t \leq t_2$ in $TQ_i \oplus T^*Q_i$ together with the interconnection condition in (5.2) is equivalent with the Lagrange-d'Alembert-Pontryagin principle in (5.4).*

Proof. It follows from the Lagrange-d'Alembert-Pontryagin principles (5.1) that one can obtain equation (5.3) as

$$\dot{q}_i = v_i \in \Delta_{Q_i}(q_i), \quad \dot{p}_i - \frac{\partial L_i}{\partial q_i} - F_i \in \Delta^\circ_{Q_i}(q_i), \quad p_i = \frac{\partial L_i}{\partial v_i}, \quad i = 1, ..., n.$$

Imposing the interconnection condition $(\dot{q}_1, ..., \dot{q}_n) \in \Delta_c(q_1, ..., q_n)$ and $F_1(q_1, v_1, p_1) \oplus \cdots \oplus F_n(q_n, v_n, p_n) \in \Delta^\circ_c(q_1, ..., q_n)$, one has

$$(\dot{q}_1, ..., \dot{q}_n) = (v_1, ..., v_n) \in \Delta(q_1, ..., q_n), \ \left(\dot{p}_1 - \frac{\partial L_1}{\partial q_1}, ..., \dot{p}_n - \frac{\partial L_n}{\partial q_n}\right) \in \Delta^\circ(q_1, ..., q_n)$$

and

$$(p_1, ..., p_2) = \left(\frac{\partial L_1}{\partial v_1}, ..., \frac{\partial L_2}{\partial v_2}\right),$$

where we note that the final distribution is given by $\Delta = (\Delta_{Q_1} \times \cdots \times \Delta_{Q_n}) \cap \Delta_c \subset TQ$ and hence that its annihilator is denoted by

$$\Delta^\circ(q_1, ..., q_n) = \Delta^\circ_{Q_1}(q_1) \times \cdots \times \Delta^\circ_{Q_n}(q_n) + \Delta^\circ_c(q_1, ..., q_n).$$

■

Theorem 5.4. *The following statements are equivalent:*

(i) A curve $(q(t), v(t), p(t))$, $t_1 \leq t \leq t_2$ in $TQ \oplus T^*Q$ satisfies the Lagrange–d'Alembert–Pontryagin principle:

$$\begin{aligned}\delta &\int_{t_1}^{t_2} \langle p(t), \dot{q}(t)\rangle - E_L(q(t), v(t), p(t))\, dt \\ &= \delta \int_{t_1}^{t_2} L(q(t), v(t)) + \langle p(t), \dot{q}(t) - v(t)\rangle\, dt = 0\end{aligned}$$

with respect to variations $\delta q \in \Delta$ and $\delta v, \delta p$ arbitrary and with fixed endpoints of $q(t)$.

(ii) Curves $(q_i(t), v_i(t), p_i(t))$, $t_1 \leq t \leq t_2$; $i = 1, ..., n$ satisfy the Lagrange–d'Alembert–Pontryagin principles

$$\begin{aligned}&\delta \int_{t_1}^{t_2} \langle p_i(t), \dot{q}_i(t)\rangle - E_L(q_i(t), v_i(t), p_i(t))\, dt + \int_{t_1}^{t_2} \langle F_i(q_i(t), v_i(t), p_i(t)), \delta q_i(t)\rangle dt \\ &= \delta \int_{t_1}^{t_2} L_i(q_i(t), v_i(t)) + \langle p_i(t), \dot{q}_i(t) - v_i(t)\rangle\, dt + \int_{t_1}^{t_2} \langle F_i(q_i(t), v_i(t), p_i(t)), \delta q_i(t)\rangle dt \\ &= 0,\end{aligned}$$

under variations $\delta q_i(t) \in \Delta_{Q_i}(q_i(t))$, arbitrary $\delta v_i(t)$ and $\delta p_i(t)$, together with $\dot{q}_i(t) \in \Delta_{Q_i}(q_i(t))$ and fixed endpoints of $q_i(t)$, such that the force vector $F(q(t), v(t), p(t)) = F_1(q_1(t), v_1(t), p_1(t)) \oplus \cdots \oplus F_n(q_n(t), v_n(t), p_n(t)) \in T^*_{q(t)}Q$ and the velocity vector $\dot{q}(t) = (\dot{q}_1(t), ..., \dot{q}_n(t)) \in T_{q(t)}Q$ satisfy

$$\dot{q}(t) \in \Delta_c(q(t)) \ \text{ and } \ F(q(t), v(t), p(t)) \in \Delta^\circ_c(q(t)),$$

where $F_i : TQ_i \oplus T^*Q_i \to T^*Q_i$ denote force fields.

Interconnected Lagrange-Dirac Dynamical Systems. Next, consider the interconnection of distinct Lagrange-Dirac dynamical systems corresponding to the above variational structures. To do this, we first consider the interconnection of distinct Dirac structures D_i on T^*Q_i, $i = 1, ..., n$, each of which is induced from given distinct distributions $\Delta_{Q_i} \subset TQ_i$. Let (E_{L_i}, D_i, X_i, F_i) be n distinct Lagrange-Dirac systems, where E_{L_i} denote the generalized energies on $TQ_i \oplus T^*Q_i$ associated to given Lagrangians L_i on TQ_i, $X_i : TQ_i \oplus T^*Q_i \to TT^*Q_i$ are the partial vector fields, and the external force fields $F_i : TQ_i \oplus T^*Q_i \to T^*Q_i$.

Recall that F_i induce the horizontal one-forms $\tilde{F}_i$ on $TQ_i \oplus T^*Q_i$ such that $\langle \tilde{F}_i, w_i \rangle = \langle F_i, T\mathrm{pr}_{Q_i}(w_i) \rangle$ for all $w_i \in T(TQ_i \oplus T^*Q_i)$, where $\mathrm{pr}_{Q_i} : TQ_i \oplus T^*Q_i \to Q_i$ are natural projections.

As before, for an interconnected system, let $Q = Q_1 \times \cdots \times Q_n$ and Δ_c be a given distribution on Q associated to the interconnection. Let $\Delta = (\Delta_{Q_1} \times \cdots \times \Delta_{Q_n}) \cap \Delta_c \subset TQ$. Then, define an interconnection Dirac structure by $D_{\mathrm{int}} = \Delta_{\mathrm{int}} \oplus \Delta^{\circ}_{\mathrm{int}}$ as in (4.2), where $\Delta_{\mathrm{int}} = (T\pi_Q)^{-1}(\Delta_c)$. Set a generalized energy $E_L = E_{L_1} + \cdots + E_{L_n} : TQ \oplus T^*Q \to \mathbb{R}$ associated to the Lagrangian $L = L_1 + \cdots + L_n : TQ \to \mathbb{R}$. Furthermore, define a partial vector field on $TQ \oplus T^*Q$ by $X = X_1 \oplus \cdots \oplus X_n : TQ \oplus T^*Q \to TT^*Q$. Define also an external force field by $F = F_1 \oplus \cdots \oplus F_n : TQ \oplus T^*Q \to T^*Q$, which induces the horizontal one-form $\tilde{F} = \tilde{F}_1 \oplus \cdots \oplus \tilde{F}_n$ on $TQ \oplus T^*Q$ such that $\langle \tilde{F}, w \rangle = \langle F, T\mathrm{pr}_Q(w) \rangle$ for all $w \in T(TQ \oplus T^*Q)$, where $\mathrm{pr}_Q : TQ \oplus T^*Q \to Q$ is the natural projection. Recall from (4.6) that the interconnection of distinct Dirac structures $D_1, ..., D_n$ through D_{int} is given by $D = (D_1 \oplus \cdots \oplus D_n) \bowtie D_{\mathrm{int}}$.

Definition 5.5. *The* ***interconnected Lagrange-Dirac dynamical system*** *is a quadruple (E_L, D, X) that satisfies*

$$(X, \mathbf{d}E_L|_{TP}) \in D,$$

where $P = \mathbb{F}L(\Delta)$. Using local coordinates $q = (q_1, ..., q_n) \in Q$, $v = (v_1, ..., v_n) \in TQ$, and $p = (p_1, ..., p_n) \in T^*_qQ$, one has

$$\dot{q} = v \in \Delta(q), \quad \dot{p} - \frac{\partial L}{\partial q} - F \in \Delta^{\circ}(q), \quad p = \frac{\partial L}{\partial v}.$$

Proposition 5.6. *The following statements are equivalent:*

(i) A curve $(q(t), v(t), p(t)) \in TQ \oplus T^*Q$, $t_1 \leq t \leq t_2$ satisfies

$$\big((\dot{q}(t), \dot{p}(t)), \mathbf{d}E_L(q(t), v(t), p(t))|_{T_{(q(t),p(t))}P}\big) \in D(q(t), p(t)),$$

where $(q(t), p(t)) = \mathbb{F}L(q(t), v(t)) \in P$.

(ii) Curves $(q_i(t), v_i(t), p_i(t))$, $t_1 \leq t \leq t_2$ satisfy

$$((\dot{q}_i(t), \dot{p}_i(t)), (\mathbf{d}E_{L_i} - \tilde{F}_i)(q_i(t), v_i(t), p_i(t))|_{T_{(q_i(t),p_i(t))}P_i}) \in D_i(q_i(t), p_i(t)),$$

where $(q_i, p_i) \in P_i = \mathbb{F}L_i(\Delta_{Q_i})$, such that

$$\dot{q}(t) = (\dot{q}_1(t), ..., \dot{q}_n(t)) \in \Delta_c(q(t)) \ \ \text{and} \ \ F(q(t), v(t), p(t)) \in \Delta^{\circ}_Q(q(t)),$$

where $\tilde{F}(q(t), v(t), p(t)) = \tilde{F}_1(q_1(t), v_1(t), p_1(t)) \oplus \cdots \oplus \tilde{F}_n(q_n(t), v_n(t), p_n(t))$.

Proof. Recall that $D = (D_1 \oplus \cdots \oplus D_n) \bowtie D_{\mathrm{int}}$, the condition

$$((\dot{q}(t), \dot{p}(t)), \mathbf{d}E(q(t), v(t), p(t))|_{T_{(q(t),p(t))}P}) \in D(q(t), p(t)), \ \text{for all } t_1 \leq t \leq t_2,$$

is equivalent to the existence of some covector

$$(F,w)=(F_1,\ldots,F_n,w_1,\ldots,w_n)\in T^*_{(q,p)}T^*Q,$$

such that

$$\left(\dot q,\dot p,-\frac{\partial L}{\partial q}-F,v+w\right)\in D_1(q_1,p_1)\oplus\cdots\oplus D_n(q_n,p_n), \tag{5.5}$$

and

$$(\dot q,\dot p,F,-w)\in D_{\mathrm{int}}(q,p). \tag{5.6}$$

It follows from the condition (5.5) that

$$\left(\dot q_i,\dot p_i,-\frac{\partial L_i}{\partial q_i}-F_i,v_i+w_i\right)\in D_i(q_i,p_i),\quad i=1,...,n,$$

and also from condition (5.6) it follows that $\dot q\in\Delta_c(q), w=0$ and $F\in\Delta_c^\circ(q)$. Furthermore, noting $\partial L/\partial q_i=\partial L_i/\partial q_i$, it follows that the curves $(q_i(t),v_i(t),p_i(t))$, $t_1\le t\le t_2$ satisfy the conditions

$$\left((\dot q_i(t),\dot p_i(t)),(\mathbf{d}E_i-\tilde F_i)(q_i(t),v_i(t),p_i(t))\big|_{T_{(q_i(t),p_i(t))}P_i}\right)\in D_i(q_i(t),p_i(t)),\quad i=1,...,n,$$

together with $(q_i(t),p_i(t))=\mathbb{F}L_i(q_i(t),v_i(t))\in P_i$. One can easily check the converse. ■

Thus, we obtain the following theorem:

Theorem 5.7. *The following statements are equivalent:*

(i) *A curve $(q,v,p)(t)$, $t_1\le t\le t_2$ in $TQ\oplus T^*Q$ satisfies*

$$((\dot q,\dot p)(t),\mathbf{d}E(q,v,p)(t)|_{TP})\in D(q(t),p(t)),$$

where $(q(t),p(t))=(q(t),(\partial L/\partial v)(t))$.

(ii) *Curves $(q_i,v_i,p_i)(t)$, $t_1\le t\le t_2$ satisfy*

$$((\dot q_i(t),\dot p_i(t)),(\mathbf{d}E_i-\tilde F_i)(q_i(t),v_i(t),p_i(t))|_{T_{(q_i(t),p_i(t))}P_i})\in D_i(q_i(t),p_i(t)),$$

where $(q_i,p_i=(q_i,\partial L_i/\partial v_i)$ and $\tilde F_i$ is the horizontal one-forms on $TQ_i\oplus T^*Q_i$ of force fields $F_i:TQ_i\oplus T^*Q_i\to T^*Q_i$, such that the condition of interconnection holds

$$F(q,v,p)(t)\in\Delta_c^\circ(q(t))\ \text{ and }\ \dot q(t)\in\Delta_c(q(t)).$$

(iii) *A curve $(q,v,p)(t)$, $t_1\le t\le t_2$ satisfies the Hamilton–Pontryagin principle:*

$$\delta\int_{t_1}^{t_2}L(q(t),v(t))+\langle p(t),\dot q(t)-v(t)\rangle dt=0$$

with respect to variations $\delta q(t)\in\Delta(q(t))$ and $\delta v,\delta p$ arbitrary, with $\dot q(t)\in\Delta(q(t))$ and with fixed endpoints of $q(t)$.

(iv) *Curves $(q_i,v_i,p_i)(t)$, $t_1\le t\le t_2$ satisfy the Lagrange–d'Alembert–Pontryagin principles:*

$$\delta\int_0^t L_i(q_i(t),v_i(t))+\langle p_i(t),\dot q_i(t)-v_i(t)\rangle dt+\int_{t_1}^{t_2}\langle F_i(t),\delta q\rangle dt=0,$$

where $\delta q_i(t)\in\Delta_{Q_i}(q_i(t))$, and $\delta v_i(t),\delta p_i(t)$ are arbitrary, $\dot q_i(t)\in\Delta_{Q_i}(q_i(t))$ and with fixed endpoints of $q(t)$, such that the interconnection condition for the force vector $F(q(t),v(t),p(t))=F_1(q_1(t),v_1(t),p_1(t))\oplus\cdots\oplus F_n(q_n(t),v_n(t),p_n(t))\in T^*_{q(t)}Q$ and the velocity vector $\dot q(t)=(\dot q_1(t),...,\dot q_n(t))\in T_{q(t)}Q$ holds as

$$\dot q(t)\in\Delta_c(q(t))\ \text{ and }\ F(q(t),v(t),p(t))\in\Delta_c^\circ(q(t)).$$

6 Examples

In this section we provide several examples of how interconnection can be carried out with the bowtie product. We have chosen rather simple examples, however the advantage of using the bowtie product is based on the procedure of tearing and interconnecting systems, which enables us to treat more complicated systems in a hierarchical structure. In future, we will seek more complex uses of the bowtie product.

(I) Example: Mass-Spring Mechanical Systems.
Let us consider an illustrative example of a mass-spring system in the context of the interconnection of Dirac structures and associated Lagrange-Dirac systems. In this example, there exists *no external force*. Let m_i and k_i be the i-th mass and spring ($i = 1, 2, 3$).

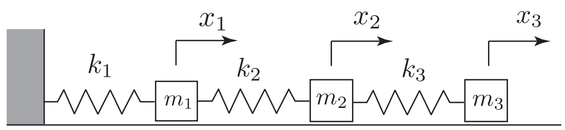

Figure 6.1: A Mass-Spring System

Tearing and Interconnection. Inspired by the concept of *tearing and interconnection* originally developed by Kron [1963], the mechanical system can be torn apart into two subsystems as in Fig 6.2, each of which can be regarded as a modular unit of the whole system. Let us call each disconnected system a "***primitive system***" by following Kron [1963]. Note that the procedure of tearing inevitably yields *interactive boundaries*[6] associated to the two disconnected subsystems, where it induces the following condition:

$$f_2 + \bar{f}_2 = 0, \qquad \dot{x}_2 = \dot{\bar{x}}_2. \tag{6.1}$$

Without the above condition, there exists no energetic interaction between the disconnected systems. In other words, *the original mechanical system can be reconstructed by interconnecting the subsystems through the continuity conditions.*

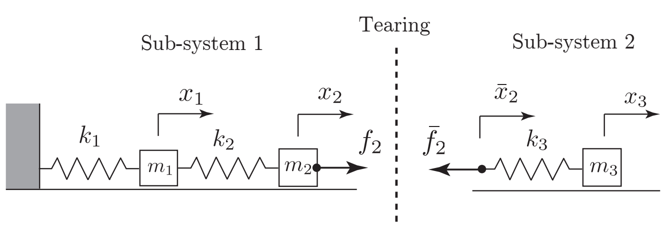

Figure 6.2: Torn-apart Systems

The continuity conditions in (6.1) imply the continuity of power flow; namely, the *power invariance* holds as

$$\mathcal{P}_2 + \bar{\mathcal{P}}_2 = 0,$$

where $\mathcal{P}_2 = \langle f_2, v_2 \rangle$ and $\bar{\mathcal{P}}_2 = \langle \bar{f}_2, \bar{v}_2 \rangle$. Needless to say, equation (6.1) can be understood as the defining condition for an interconnection Dirac structure.

[6]Such an interactive boundary is called a "port" in circuit theory (see, for instance, Chua, Desoer and Kuh [1987]).

Primitive Lagrangian Systems. Let us consider how dynamics of the disconnected subsystems can be formulated in the context of Lagrange-Dirac dynamical systems.

The configuration space of the subsystem 1 may be given by $Q_1 = \mathbb{R} \times \mathbb{R}$ with local coordinates (x_1, x_2), while the configuration space of the subsystem 2 is $Q_2 = \mathbb{R} \times \mathbb{R}$ with local coordinates $(\bar{x}_2, x_3)$. We can naturally define the canonical Dirac structures $D_1 \in \daleth(T^*Q_1)$ and $D_2 \in \daleth(T^*Q_2)$ in this example. For Primitive System 1, the Lagrangian $L_1 : TQ_1 \to \mathbb{R}$ is given by

$$L_1(x_1, x_2, v_1, v_2) = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 - \frac{1}{2} k_1 x_1^2 - \frac{1}{2} k_2 (x_2 - x_1)^2,$$

while the Lagrangian $L_2 : TQ_2 \to \mathbb{R}$ for Primitive System 2 is given by

$$L_2(\bar{x}_2, x_3, \bar{v}_2, v_3) = \frac{1}{2} m_3 v_3^2 - \frac{1}{2} k_3 (x_3 - \bar{x}_2)^2.$$

Then, we can define the generalized energy E_1 on $TQ_1 \oplus T^*Q_1$ by $E_1(x_1, x_2, v_1, v_2, p_1, p_2) = p_1 v_1 + p_2 v_2 - L_1(x_1, x_2, v_1, v_2)$ and also define the generalized energy E_2 on $TQ_2 \oplus T^*Q_2$ by $E_2(\bar{x}_2, x_3, \bar{v}_2, v_3, \bar{p}_2, p_3) = \bar{p}_2 \bar{v}_2 + p_3 v_3 - L_2(\bar{x}_2, x_3, \bar{v}_2, v_3)$.

Though the original system has no external force, each disconnected primitive system has an interconnection constraint force at the interactive boundary. When viewing each system separately, the constraint force acts as an external force. This is because *tearing always yields constraint forces at the boundaries associated with the disconnected primitive systems*, as shown in Fig. 6.2

The ***constraint force fields*** at the boundaries $F_{c1} : TQ_1 \oplus T^*Q_1 \to T^*T^*Q_1$ and $F_{c2} : TQ_2 \oplus T^*Q_2 \to T^*T^*Q_2$ are given by $F_{c1} = (x_1, x_2, v_1, v_{2,}, p_1, p_2, 0, -f_2, 0, 0)$ an $F_{c2} = (\bar{x}_2, x_3, \bar{v}_2, v_3, \bar{p}_2, p_3, -\bar{f}_2, 0, 0, 0)$. Further, let $X_1 : TQ_1 \oplus T^*Q_1 \to TT^*Q_1$ be the partial vector field, which is defined at each point $(p_1 = m_1 v_1, p_2 = m_2 v_2) \in P_1 = \mathbb{F}L(TQ_1)$ as $X_1(x_1, x_2, v_1, v_2, p_1, p_2) = (x_1, x_2, p_1, p_2, \dot{x}_1, \dot{x}_2, \dot{p}_1, \dot{p}_2)$. Similarly, let $X_2 : TQ_2 \oplus T^*Q_2 \to TT^*Q_2$ be the partial vector field, which is defined at each point $(\bar{p}_2 = 0, p_3 = m_3 v_3) \in P_2 = \mathbb{F}L(TQ_2)$ as $X_2(\bar{x}_2, x_3, \bar{v}_2, v_3, \bar{p}_2, p_3) = (\bar{x}_2, x_3, \bar{p}_2, p_3, \dot{\bar{x}}_2, \dot{x}_3, \dot{\bar{p}}_2, \dot{p}_3)$, together with the *consistency condition* $\dot{\bar{p}}_2 = 0$.

Primitive System 1: We can formulate dynamics of Primitive System 1 in the context of the Lagrange-Dirac dynamical system (E_1, D_1, X_1, F_1) as

$$(X_1, \mathbf{d}E_1|_{TP_1} - \tilde{F}_{c1}) \in D_1,$$

which may be given in coordinates by

$$\dot{x}_1 = v_1, \quad \dot{x}_2 = v_2, \quad \dot{p}_1 = -k_1 x_1 - k_2(x_1 - x_2), \quad \dot{p}_2 = k_2(x_1 - x_2) + f_2,$$

and with $p_1 = m_1 v_1$ and $p_2 = m_2 v_2$.

Primitive System 2: Similarly, we can also formulate dynamics of Primitive System 2 in the context of the Lagrange-Dirac dynamical system (E_2, D_2, X_2, F_2) as

$$(X_2, \mathbf{d}E_2|_{TP_2} - \tilde{F}_{c2}) \in D_2,$$

which may be given in coordinates by

$$\dot{\bar{x}}_2 = \bar{v}_2, \quad \dot{x}_3 = v_3, \quad \dot{\bar{p}}_2 = k_3(x_3 - \bar{x}_2) + \bar{f}_2, \quad \dot{p}_3 = -k_3(x_3 - \bar{x}_2),$$

together with $\bar{p}_2 = 0$ and $p_3 = m_3 v_3$ as well as $\dot{\bar{p}}_2 = 0$.

Recall that each primitive system is physically independent of the other, which implies that there exists no energetic interaction between them. Next, let us see how the disconnected primitive systems can be interconnected through a Dirac structure.

Interconnection of Distinct Dirac Structures. Let $Q = Q_1 \times Q_2 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ be an ***extended configuration space*** with local coordinates $x = (x_1, x_2, \bar{x}_2, x_3)$. Recall that the Dirac sum of the Dirac structures is given by $D_1 \oplus D_2$ on T^*Q. Now, the constraint distribution due to the interconnection is given by

$$\Delta_c(x) = \{v \in T_xQ \mid \langle \omega_Q(x), v\rangle = 0\},$$

where $\omega_Q = dx_2 - d\bar{x}_2$ is a one-form on Q. On the other hand, the annihilator $\Delta_c^\circ \subset T^*Q$ is defined by

$$\Delta_c^\circ(x) = \{f = (f_1, f_2, \bar{f}_2, f_3) \in T_x^*Q \mid \ \langle f, v\rangle = 0 \text{ and } v \in \Delta_c(x)\}.$$

It follows from this codistribution that $f_2 = -\bar{f}_2$, $f_1 = 0$ and $f_3 = 0$. Hence, we obtain the conditions for the interconnection as in equation (6.1); namely, $f_2 + \bar{f}_2 = 0$ and $v_2 = \bar{v}_2$. Recall that the distribution Δ_{int} on T^*Q is obtained by $\Delta_{\mathrm{int}} = (T\pi_Q)^{-1}(\Delta_c) \subset TT^*Q$, and hence the Dirac structure D_{int} can be defined as in (4.2). Further, recall that the interconnected Dirac structure D on T^*Q is given by

$$D = (D_1 \oplus D_2) \bowtie D_{\mathrm{int}}.$$

Interconnected Lagrange-Dirac Systems. Now, let us see how two primitive Lagrange-Dirac systems, namely, (E_{L_1}, D_1, X_1, F_1) and (E_{L_2}, D_2, X_2, F_2) can be interconnected to be a Lagrange-Dirac dynamical system.

Define the Lagrangian $L : TQ \to \mathbb{R}$ for the interconnected system by $L = L_1 + L_2$, and hence the generalized energy is given by $E_L = E_{L_1} + E_{L_2} : TQ \oplus T^*Q \to \mathbb{R}$. Let $\Delta = (TQ_1 \times TQ_2) \cap \Delta_{\mathrm{int}}$. Set a partial vector field by $X = X_1 \oplus X_2 : TQ \oplus T^*Q \to TT^*Q$, which is defined at points $(x_1, x_2, \bar{x}_2, x_3, v_1, v_2, \bar{v}_2, v_3, p_1 = m_1v_1, p_2 = m_2v_2, \bar{p}_2 = 0, p_3 = m_3v_3) \in P = \mathbb{F}L(\Delta)$, where the consistency condition $\dot{\bar{p}}_2 = 0$ holds.

Finally, the ***interconnected Lagrange-Dirac system*** is given by (E_L, D, X), which satisfies, for each (x, v, p),

$$(X(x, v, p), \mathbf{d}E_L(x, v, p)|_{TP}) \in D(x, p),$$

and with $(x, p) \in P = \mathbb{F}L(\Delta)$.

The Lagrange-d'Alembert-Pontryagin Principle. Alternatively, we can develop the interconnected Lagrange-Dirac system by the Lagrange-d'Alembert-Pontryagin principle:

$$\begin{aligned}\delta \int_a^b L_1(x_1, x_2, v_1, v_2) + p_1(\dot{x}_1 - v_1) + p_2(\dot{x}_2 - v_2) \\ + L_2(\bar{x}_2, x_3, v_3) + \bar{p}_2(\dot{\bar{x}}_2 - \bar{v}_2) + p_3(\dot{x}_3 - v_3)\, dt,\end{aligned}$$

for all $\delta x_2 = \delta\bar{x}_2$, for all δv and δp, and with $v_2 = \bar{v}_2$.

Thus, we can obtain the dynamics of the interconnected Lagrange-Dirac system.

(II) Example: Electric Circuits

Consider the electric circuit depicted in Figure 6.3, where R denotes a resistor, L an inductor, and C a capacitor.

As in Figure 6.4, we decompose the circuit into the disconnected two circuits, each of which we shall call "Primitive Circuit 1" and "Primitive Circuit 2", in which S_1 and S_2 denote external ports. In order to reconstruct the original circuit in Figure 6.3, the external ports may be connected by equating currents across each.

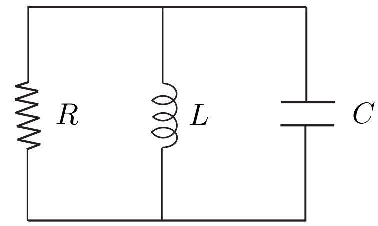

Figure 6.3: R-L-C Circuit

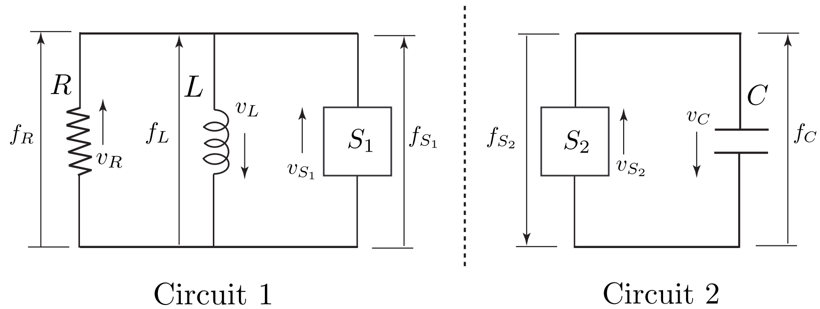

Figure 6.4: Disconnected Circuits

Primitive Circuit 1. The configuration manifold for circuit 1 is denoted by $Q_1 = \mathbb{R}^3$ with local coordinates $q_1 = (q_R, q_L, q_{S_1})$, where q_R, q_L and q_{S_1} are the charge fluxes through the resistor R, inductor L and port S_1 respectively. There exists the KCL constraint distribution $\Delta_{Q_1} \subset TQ_1$ in Circuit 1, which is given by for each $q_1 = (q_R, q_L, q_{S_1}) \in Q_1$,

$$\Delta_{Q_1}(q_1) = \{v_1 = (v_R, v_L, v_{S_1}) \in T_{q_1}Q_1 \mid v_R - v_L + v_{S_1} = 0\},$$

where $v_1 = (v_R, v_L, v_{S_1})$ denotes the current vector at each q_1, while the KVL constraint is given by its annihilator $\Delta^{\circ}_{Q_1}$. Then, we can naturally define the induced Dirac structure D_1 on T^*Q_1 from Δ_{Q_1} as before.

The Lagrangian for Circuit 1, namely, $\mathcal{L}_1$ on TQ_1 is given by

$$\mathcal{L}_1(q_1, v_1) = \frac{1}{2} L_1 v_L^2,$$

which is degenerate. Define the generalized energy $E_{\mathcal{L}_1}$ on $TQ_1 \oplus T^*Q_1$ by $E_{\mathcal{L}_1}(q_1, v_1, p_1) = \langle p_1, v_1 \rangle - \mathcal{L}_1(q_1, v_1)$ on $TQ_1 \oplus T^*Q_1$. Circuit 1 also has the external force field due to the resistor and the external port, $F_1 : TQ_1 \oplus T^*Q_1 \to \mathbb{R}$ as

$$F_1(q_1, v_1, p_1) = (-Rv_R)dq_R + f_{S_1} dq_{S_1}.$$

So, we can set up the Lagrange-Dirac dynamical system $(E_{\mathcal{L}_1}, D_1, X_1, F_1)$.

Primitive Circuit 2. The configuration manifold for Circuit 2 is $Q_2 = \mathbb{R}^2$ with local coordinates $q_2 = (q_{S_2}, q_C)$, where q_{S_2} is the charge flux through the port S_2 and q_C is the charge stored in the capacitor. The KCL constraint distribution is given by, for each q_2,

$$\Delta_2(q_2) = \{v_2 = (v_{S_2}, v_C) \in T_{q_2}Q_2 \mid v_C - v_{S_2} = 0\}$$

and so the KVL space is given by its annihilator $\Delta_2^{\circ}(q_2)$. This gives us the Dirac structure D_2 on T^*Q_2. Set the Lagrangian $\mathcal{L}_2 : TQ_2 \to \mathbb{R}$ for Circuit 2 as

$$\mathcal{L}_2 = \frac{1}{2C} q_C^2$$

and set the generalized energy $E_{\mathcal{L}_2}(q_2, v_2, p_2) = \langle p_2, v_2 \rangle - \mathcal{L}_2(q_2, v_2)$ on $TQ_2 \oplus T^*Q_2$. Circuit 2 has the external force field due to the resistor and the external port S_2, namely, $F_2 : TQ_2 \oplus T^*Q_2 \to \mathbb{R}$ as

$$F_2(q_2, v_2, p_2) = -f_{S_2} dq_{S_2}.$$

Then, we can formulate the Lagrange-Dirac dynamical system $(E_{\mathcal{L}_2}, D_2, X_2, F_2)$.

The Interconnection Dirac Structure Set $Q = Q_1 \times Q_2$. The interconnection constraint is given by

$$\Delta_c = \{v = (v_1, v_2) \in TQ \mid v_{S_1} = v_{S_2}\}$$

By the tangent lift $\Delta_{\rm int} = T\pi_Q^{-1}(\Delta_{12})$, we can define the interconnection Dirac structure as

$$D_{\rm int} = \Delta_{\rm int} \oplus \Delta_{\rm int}^{\circ},$$

which is denoted, in local coordinates, by

$$\begin{aligned} D_{\rm int} = \{(\dot{q}_1, \dot{q}_2, \dot{p}_1, \dot{p}_2, \alpha_1, \alpha_2, w_1, w_2) \in TT^*Q \oplus T^*T^*Q \mid \\ \dot{q}_{S_1} = \dot{q}_{S_2} = 0,\ w_1 = 0,\ w_2 = 0,\ \alpha_1 + \alpha_2 \in \mathrm{span}(dq_{S_1} - dq_{S_2})\}. \end{aligned}$$

The Interconnected System. Now, we can define the interconnected Dirac structure by

$$D = (D_1 \oplus D_2) \bowtie D_{\rm int},$$

which may be viewed as the induced Dirac structure from the constraint space

$$\Delta = (\Delta_1 \times \Delta_2) \cap \Delta_c,$$

which is given, in coordinates $(q_1, q_2) = (q_R, q_L, q_{S_1}, q_{S_2}, q_C)$, by

$$\Delta(q_1, q_2) = \{(v_1, v_2) = (v_R, v_L, v_{S_1}, v_{S_2}, v_C) \mid v_R - v_L + v_C = 0,\ v_{S_1} = v_{S_2},\ v_{S_1} = v_C\}.$$

Set the Lagrangian for the interconnected system as $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$ and the external force field $F = F_1 \oplus F_2 : TQ \oplus T^*Q \to T^*Q$. Set also $E_{\mathcal{L}} = E_{\mathcal{L}_1} + E_{\mathcal{L}_2}$. Let $X = X_1 \oplus X_2 : TQ \oplus T^*Q \to TT^*Q$ be a partial vector field. The interconnected Lagrange-Dirac dynamical system is given by the quadruple $(E_{\mathcal{L}}, D, X, F)$, which satisfies

$$(X, (\mathbf{d}E_{\mathcal{L}} - \tilde{F})|_{TP}) \in D,$$

where $P = \mathbb{F}L(\Delta)$.

(III) Example: A Rolling Ball on Rotating Tables

Consider the mechanical system depicted in Fig. 6.5, where there are rotating tables via a gear and a ball is rolling on one of the tables without slipping. We assume that there is no loss of energy and non-slipping in the gear and also that the external torque is constant. Let I_i be moments of inertia for the table $i = 1, 2$. Let us decompose the system into distinct three systems; (1) a rotating (small) table, (2) a rotating (large) table, and (3) a rolling ball on a table.

Primitive System 1. For Primitive System 1 the configuration manifold is the circle, $Q_1 = S^1$. The Lagrangian is the rotational energy of the system.

$$L_1(s_1, \dot{s}_1) = \frac{I_1}{2} \dot{s}_1^2,$$

and set the generalized energy E_{L_1} on $TQ_1 \oplus T^*Q_1$. We employ the canonical Dirac structure on T^*Q_1 as

$$D_1 = \{(\dot{s}_1, \dot{p}_{s_1}, \alpha_{s_1}, w_{s_1}) \mid \dot{s}_1 = w_{s_1}, \dot{p}_{s_1} + \alpha_{s_1} = 0\}$$

There is a constant torque acting on Primitive System 1, given by a map $F_1 : TQ_1 \oplus T^*Q_1 \to T^*Q_1$ defined as $F(s_1, v_{s_1}, p_{s_1}) = \tau ds_1$, where τ is constant in $\mathbb{R}$.

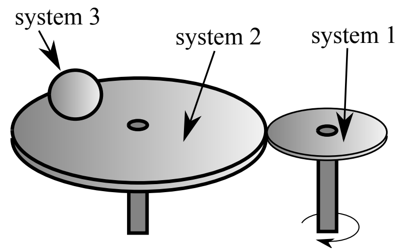

Figure 6.5: A Rolling Ball on Rotating Tables without Slipping

Primitive System 2. For Primitive System 2, the configuration manifold is also the circle, $Q_2 = S^1$ and the Lagrangian is again the rotational energy

$$L_2(s_2, \dot{s}_2) = \frac{I_2}{2}\dot{s}_2^2$$

and set the generalized energy E_{L_2} on $TQ_2 \oplus T^*Q_2$. Again, we have the canonical Dirac structure

$$D_2 = \{(\dot{s}_2, \dot{p}_{s_2}, \alpha_{s_2}, w_{s_2}) \mid \dot{s}_2 = w_{s_2}, \dot{p}_{s_2} + \alpha_{s_2} = 0\}.$$

Primitive System 3. Primitive System 3 is a rolling sphere of uniform density and radius 1.The sphere moves in space by changing its position and orientation relative to a reference configuration. The configuration manifold is given by the special Euclidean group $Q_3 = \mathrm{SE}(3)$, which we parametrize as (R, u) where $R \in \mathrm{SO}(3), u \in \mathbb{R}^3$. Following Marsden and Ratiu [1999], let β be the set of points of the sphere in the reference configuration. For configuration $(R, u) \in Q_3$, a point $x \in \beta$ is transformed into $\mathbb{R}^3$ by the action $(R, u) \cdot x = (R \cdot x) + u$. The Lagrangian is given by the kinetic energy as

$$L_3(R, u, \dot{R}, \dot{u}) = \int_\beta \frac{\rho}{2}\|\dot{R}x + \dot{u}\|^2 dx,$$

where $\|\dot{R}x + \dot{u}\|^2 = x^T\dot{R}^T\dot{R}x + 2x^T\dot{R}^T\dot{u} + \dot{u}^2$. We use body coordinates such that the center of the sphere in the reference configuration is at the origin so that $\int_\beta x dx = 0$. Substituting these relations, the above Lagrangian is to be

$$L_3 = \int_\beta \frac{\rho}{2}\left(x^T\dot{R}^T\dot{R}x + \dot{u}^2\right) dx.$$

Setting $m_3 = \int_\beta \rho dx = \frac{4}{3}\pi\rho$ and noting that $\int_\beta x_i x_j dx = 0$ when $i \neq j$, one finally obtains

$$L_3 = \frac{m_3}{2}\left(\mathrm{tr}(\dot{R}^T\dot{R}) + \dot{u}^2\right)$$

and with the generalized energy E_{L_3} on $TQ_3 \oplus T^*Q_3$.

Since the motion along the z-direction is constrained so that the ball does not leave the plane of table 2, we get the holonomic constraint

$$\Delta_3 = \{(\dot{R}, \dot{u}) \mid \dot{u}_3 = 0\}.$$

This yields the induced Dirac structure

$$\begin{aligned} D_3 = \{&(\delta R, \delta u, \delta p_R, \delta p_u, \alpha_R, \alpha_u, w_R, w_u) \in TT^*Q_3 \oplus T^*T^*Q_3 \mid \\ &\delta u_3 = 0, \delta u = w_u, \delta R = w_R,\ \delta p_R + \alpha_R = 0,\ \delta p_u + \alpha_u = \lambda\mathbf{d}z \text{ for some } \lambda \in \mathbb{R}\}. \end{aligned}$$

Interconnection Constraints. Let $Q = Q_1 \times Q_2 \times Q_3$. In order to interconnection the three primitive systems, we need to impose the constraints due to the non slip conditions. First, consider the mapping $\Phi(v_s) = s^{-1}v_s$ for $v_s \in T_sS^1$. The map Φ sends the tangent fiber T_sS^1 to the Lie-algebra $\mathfrak{s} = T_eS^1 \approx \mathbb{R}$, where e is the identity element on S^1 viewed as a group. From this point on, we do not explicitly invoke Φ but use it to interpret TS^1 as $S^1 \times \mathbb{R}$. The interconnection constraint between System 1 and System 2 is given by

$$\Delta_{c,1} = \{(\dot{s}_1, \dot{s}_2, \dot{R}, \dot{u}) \in TQ \mid \dot{s}_1 + \dot{s}_2 = 0\}$$

and with its annihilator

$$\Delta^\circ_{c,1} = \mathrm{span}(\omega_1)$$

where $\omega_1 = ds_1 - ds_2$. This constraint ensures that the gears rotate (without slipping) at the same speed in opposite directions.

Next, we consider the interconnection constraint between Systems 2 and 3. Note that the velocity of a point located at the bottom of the sphere is given by

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \dot{R}R^T \cdot \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} + \dot{u}.$$

Note also that a point rotating on the gear of system 2 with the axle taken to be the origin has velocity

$$\begin{pmatrix} v'_1 \\ v'_2 \end{pmatrix} = \begin{pmatrix} 0 & -\dot{s}_2 \\ \dot{s}_2 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}.$$

So the non slip condition between System 2 and 3 is given by

$$\Delta_{c,2} = \{(\dot{s}_1, \dot{s}_2, \dot{R}, \dot{u}) \in TQ \mid \ \mathbf{i} \cdot (-\dot{R}R^T \cdot \mathbf{k} + \dot{u}) = -\dot{s}_2 \cdot u_2, \ \mathbf{j} \cdot (-\dot{R}R^T \cdot \mathbf{k} + \dot{w}) = \dot{s}_2 \cdot u_1\},$$

where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the basis on $\mathbb{R}^3$. Setting the interconnection constraint distribution

$$\Delta_c = \Delta_{c,1} \cap \Delta_{c,2},$$

let $\Delta_{\mathrm{int}} := (T\pi_Q)^{-1}(\Delta_c)$ and with its annihilator $\Delta^\circ_{\mathrm{int}}$. Then, the interconnection Dirac structure is given by

$$D_{\mathrm{int}} = \Delta_{\mathrm{int}} \oplus \Delta^\circ_{\mathrm{int}}.$$

The Interconnected Lagrange-Dirac System. The Dirac structure for the interconnected system is given by

$$D = (D_1 \oplus D_2 \oplus D_3) \bowtie D_{\mathrm{int}}$$

and we use the Lagrangian $L = L_1 + L_2 + L_3$. The dynamics of the system are implied by the constraint:

$$(X, \mathbf{d}E_L|_{TP}) \in D,$$

where $X : TT^*Q \oplus T^*T^*Q \to TT^*Q$ is a partial vector field, and $E_L : TQ \oplus T^*Q \to \mathbb{R}$ is the generalized energy of L. This gives us the interconnected Lagrangian system (E_L, D, X). By Theorem 4.9, these dynamics use the constraint distribution

$$\Delta = (TQ_1 \oplus TQ_2 \oplus \Delta_3) \cap \Delta_c,$$

and the canonical two-form on T^*Q restricted to Δ as the Dirac two-form. Noting the annihilator is given by

$$\Delta^\circ = \Delta^\circ_3 + \Delta^\circ_{c,1} + \Delta^\circ_{c,2},$$

we finally obtain the dynamics as

$$\dot{q} = v \in \Delta, \quad \dot{p} - \frac{\partial L}{\partial q} \in \Delta^\circ, \quad p = \frac{\partial L}{\partial v}.$$

7 Conclusions and Future Works

In this paper, we provided a notion of interconnection for Dirac structures and their associated Lagrange-Dirac dynamical systems. The principal idea lies in the fact that the *interconnection of distinct Dirac structures itself is a Dirac structure*. We also clarified the interconnection of the associated Lagrange-Dirac dynamical systems with the variational structure by the Lagrange-d'Alembert-Pontryagin principle. To do this, we developed the bowtie product, $\bowtie$, which serves us well in the case of interconnection by constraints. We then showed how the bowtie product can effectively carry the Lagrange-d'Alembert-Pontryagin principle to the interconnected system. Lastly, we demonstrated our theory with some illustrative examples of a mass-spring mechanical system, an interconnection in circuits, and a non-holonomic mechanical system.

We raise the following topics as arena's for further investigation:

- Applications to complicated systems such as guiding central motion problems, multibody systems, fluid-rigid body interactions, passivity controlled interconnected systems, etc. (see, for example, Littlejohn [1983]; Featherstone [1987]; Yoshimura [1995]; van der Schaft [1996] and Ortega, van der Schaft, Maschke and Escobar [2002]).
- Reduction and symmetry for interconnected Lagrange-Dirac systems (Yoshimura and Marsden [2007b], Yoshimura and Marsden [2009]).
- The use of more general interconnection Dirac structures with nontrivial Dirac two-forms such as the one given with symplectic gyrators (in this paper, we restrict D_{int} to the form $\Delta_{\mathrm{int}} \oplus \Delta^{\circ}_{\mathrm{int}}$). As to examples of such general interconnections in physical systems, see for instance Wyatt and Chua [1977]; Yoshimura [1995] and Maschke, van der Schaft and Breedveld [1995].
- The integrability condition for the bowtie product of Dirac structures. As to the integrability condition for Dirac structures, see Dorfman [1993] and Dalsmo and van der Schaft [1998]. The link with *symplectic categories* (see Weinstein [2009]).
- Discrete version of interconnected Lagrange-Dirac mechanics; namely, discretizing the Hamilton-Pontryagin principle one arrives at a discrete mechanics version of Dirac structures (see Bou-Rabee and Marsden [2009] and Leok and Ohsawa [2010]).

A Appendix: The Bowtie Product $\bowtie$ and the Dirac Tensor Product $\boxtimes$

We will show the equivalence between the bowtie product $\bowtie$ and the Dirac tensor product $\boxtimes$ developed by Gualtieri [2007], where the definition of the tensor product $\boxtimes$ is given as follows.

Definition A.1. *Let $D_a, D_b \in \daleth(P)$. Let $d : P \hookrightarrow P \times P$ be the diagonal embedding in $P \times P$. The **tensor product** of D_a and D_b is defined as*

$$D_a \boxtimes D_b := d^*(D_a \oplus D_b) = \frac{(D_a \oplus D_b) \cap (K^{\perp} + K)}{K},$$

where $K = \{(0,0)\} \oplus \{(-\xi, \xi) \mid \xi \in T^*P\} \subset T(P \times P) \oplus T^*(P \times P)$ and $K^{\perp}$ is the orthogonal complement in $T(P \times P) \oplus T^*(P \times P)$.

Proposition A.2. *The bowtie product $\bowtie$ and the tensor product $\boxtimes$ of D_a and D_b are equivalent:*

$$D_a \boxtimes D_b \equiv D_a \bowtie D_b,$$

where the notation $\equiv$ implies the fact that these structures are isomorphic.

Proof. First note that

$$K^{\perp} = \{(v, v)\} \oplus T^*(P \times P)$$

so that $K + K^{\perp} = \{(v, v)\} \oplus T^*(P \times P)$. Then we see that

$$(D_a \oplus D_b) \cap (K + K^{\perp}) = \{(v, v, \alpha_A, \alpha_B) \in T(P \times P) \oplus T^*(P \times P) \mid (v, v, \alpha_A, \alpha_B) \in D_a \oplus D_b\}$$

and the equivalence class $D_a \boxtimes D_b$ is

$$D_a \boxtimes D_b \approx \{(v, v, \alpha, 0) \mid \exists \beta \text{ such that } (v, v, \alpha + \beta, -\beta) \in D_a \oplus D_b\},$$

where we chose the element $(v, v, \alpha, 0)$ as the representative for the equivalence class

$$[(v, v, \alpha, \alpha')] = \{(v, v, \alpha + \xi, \alpha - \xi), \forall \xi \in T^*P\} \in D_a \boxtimes D_b.$$

Splitting the expression $D_a \oplus D_b$ into D_a and D_b, we obtain

$$D_a \boxtimes D_b = \{(v, v, \alpha, 0) \in T(P \times P) \oplus T^*(P \times P) \mid \exists \beta \text{ such that } (v, \alpha + \beta) \in D_a, (v, -\beta) \in D_b\}$$

we see that the assignment $\phi : (v, v, \alpha, 0) \mapsto (v, \alpha)$ is a bijection and also that

$$\phi(D_a \boxtimes D_b) = \{(v, \alpha) \in TP \oplus T^*P \mid \exists \beta \text{ such that } (v, \alpha + \beta) \in D_a, (v, -\beta) \in D_b\}$$

this is just the definition of $D_a \bowtie D_b$. ∎

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